Sequences of singularly perturbed functionals generating free-discontinuity problems

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Abstract
We prove that a wide class of singularly perturbed functionals generates as $\Gamma$-limit a functional related to a free-discontinuity problem. Several applications of the result are shown.

Key words: free-discontinuity problems, singular perturbations, $\Gamma$-convergence

1 Introduction
Many problems arising from Fracture Mechanics and Computer Vision lead to free-discontinuity problems, that is to the minimization of functionals defined in spaces of discontinuous functions (namely $BV$ and $SBV$) involving energies with a bulk part and a surface part concentrated along the (free) discontinuity zone. This paper is concerned with the variational approximation in the sense of De Giorgi’s $\Gamma$-convergence of such energies by smooth functionals defined on Sobolev spaces. In the last years a lot of work has been addressed in this direction in view of numerical applications and in view of defining an evolution model (as limit of the gradient flows of the approximating functionals): for a general survey on free-discontinuity problems and their approximation we refer to [10] and [6].

When the volume part of the energy is given by $\int_{\Omega} |\nabla u|^2 \, dx$, heuristic considerations suggest to use, as approximating functionals, energies of the form

$$\frac{1}{\varepsilon} \int_{\Omega} f(\sqrt{\varepsilon} |\nabla u|) \, dx,$$

where $f : [0, +\infty) \to [0, +\infty)$ is quadratic near the origin and with finite limit at infinity. However, an easy convexity argument shows that energies of this kind $\Gamma$-converge to the zero functional. Various methods have been developed to bypass this convexity constraint, most of them exploiting the De Giorgi’s suggestion of replacing the functionals above with suitable non-local versions (see [13], [21], [17]). The approach we consider here is based on singular perturbations and consists in adding a “small” term depending on higher derivatives: the idea is to impose a bound on the oscillations of minimizing sequences by penalizing abrupt changes of the gradient. So we are led to consider energies of the form

$$\frac{1}{\varepsilon} \int_{\Omega} f(\sqrt{\varepsilon} |\nabla u|) \, dx + r(\varepsilon) \int_{\Omega} \|\nabla^2 u\|^2 \, dx, \quad (1.1)$$

where $r(\varepsilon)$ is a function which vanishes as $\varepsilon \to 0^+$.

The first step in this direction was taken by Alicandro, Braides & Gelli in [2]: they showed that the one dimensional functionals

$$\frac{1}{\varepsilon} \int_{0}^{1} f(\sqrt{\varepsilon} |u'|) \, dx + \varepsilon^3 \int_{0}^{1} |u''|^2 \, dx,$$
with \( f(t) = \alpha t^2 \wedge \beta \), \( \Gamma \)-converge with respect to the \( L^1 \)-norm to the functional

\[
\alpha \int_0^1 |u'|^2 \, dx + \sum_{S_u} c(\beta) \sqrt{u^+ - u^-} \, ,
\]

where \( c(\beta) \) is a constant depending on \( \beta \); later Alicandro & Gelli treated the \( N \)-dimensional case (see [3]). We aim to extend the results above to general functionals of the form (1.1), where \( f \) is still quadratic near the origin, but possibly unbounded. In fact we face the problem in a more general framework, by investigating the asymptotic behaviour of

\[
F_\varepsilon(u) := \int_\Omega f_\varepsilon(|\nabla u|) \, dx + (r(\varepsilon))^3 \int_\Omega \|\nabla^2 u\|^2 \, dx ,
\]

where \( f_\varepsilon \) is any family of positive non-decreasing functions with a convex or convex-concave shape (i.e. there exists \( x_\varepsilon > 0 \) such that \( f_\varepsilon \) is convex in \( [0, x_\varepsilon] \) and concave in \( [x_\varepsilon, +\infty) \)); let us remark that such a structure assumption is quite natural for this kind of problems (see, for example, [14], [15], [22]). In the main theorem of the paper (Theorem 3.2) we prove that the \( \Gamma \)-limits of (1.2) are related to the pointwise limits of \( f_\varepsilon(t) \) and of \( r(\varepsilon)f_\varepsilon(t/r(\varepsilon)) \); if for an infinitesimal subsequence \( (\varepsilon_n) \) we have

a) \( f_\varepsilon \to g \) pointwise,  
b) \( r(\varepsilon_n)f_\varepsilon(\cdot/r(\varepsilon_n)) \to b \) pointwise,

then \( (F_\varepsilon_n) \) \( \Gamma \)-converges to a functional \( F \) defined on \( BV(\Omega) \) and taking the form

\[
F(u) = \int_\Omega f(\nabla u) \, dx + \int_{S_u} \varphi(\mathcal{D}^\varepsilon u) \, C|\mathcal{D}^\varepsilon u| ,
\]

where \( C \) (possibly equal to \( +\infty \), meaning that \( F \) is finite only on \( SBV \)), \( f \), and \( \varphi \) can be characterized in terms of \( g \) and \( b \).

The “regularizing” effect due to the presence of the second derivatives in the approximating functionals, determines a restriction on the regularity and on the growth of the jump-function \( \varphi \), which turns out to satisfy the growth condition

\[
C_1(\sqrt{z} - 1) \leq \varphi(z) \leq C_2(z + 1) \quad \forall z \geq 0 ,
\]

for suitable \( C_1, C_2 > 0 \); whenever \( \lim_{t \to 0} b(t)/t \neq 0 \); moreover, we always have \( \varphi(0) = 0 \). In particular, the Mumford-Shah functional is not reachable by our procedure. However, since for any positive, convex, and superlinear function \( g \) and for any positive and concave function \( b \) with \( \lim_{t \to +\infty} b(t)/t = +\infty \), it is possible to construct a family \( (f_\varepsilon) \) and a rescaling function \( r(\varepsilon) \) such that conditions a) and b) above are fulfilled, we see that a wide class of free-discontinuity functionals with \( \varphi \) satisfying (3.57) can be approximated. Letting \( b \) vary among the possible choices, we may conjecture to recover most of the admissible asymptotic behaviours as the following fact seems to suggest: all the functions of the form \( \varphi(t) = ct^\gamma \), with \( c > 0 \) and \( \gamma \) varying in \( [1/2, 1] \) are reachable, and for every \( \gamma \in (1/2, 1) \) a function \( \varphi \) can be generated such that

\[
\lim_{z \to +\infty} \frac{\varphi(z)}{z^\gamma} = +\infty \quad \text{and} \quad \lim_{z \to +\infty} \frac{\varphi(z)}{z^{\gamma+\varepsilon}} = 0 \quad \forall \varepsilon > 0 .
\]

As announced, in Section 4 we apply our theorem to prove that if \( f \) is quadratic near the origin, sublinear, and concave at infinity, there exists a rescaling function \( r(\varepsilon) \) (explicitly given in terms of \( f \)) such that the family (1.1) \( \Gamma \)-converges, up to passing to a subsequence, to a free-discontinuity functional like (1.3). All the possible \( \Gamma \)-limits of that family are classified. The rescaling \( r(\varepsilon) \) is unique up to asymptotic equivalence, in the sense that when we use functions with a different behaviour near the origin, we obtain in the limit either \( F \equiv 0 \) or the functional \( \alpha \int_\Omega |\nabla u|^2 \, dx \) defined only on \( H^1(\Omega) \).

In a recent paper ([9]) Bouchitté, Dubs & Seppecher considered the one-dimensional functionals

\[
F_\varepsilon(u) := \int_I \frac{|u'|}{1 + (\varepsilon |u'|)^p} \, dx + \varepsilon^{2p} \int_I |u''|^2 \, dx
\]
defined in $W^{2,2}(I)$ and proved that they $\Gamma$-converge to the functional $F$ (defined in $SBV(I)$) given by

$$F(u) := \int_I |u'|^2 \, dx + k_p \sum_{x \in \mathbb{S}_{\mathbb{R}}^+} (u^+ - u^-)^\frac{p}{2} \, \nu.$$ 

When $p \leq 2$ their result is a particular case of ours (but it is proved by the use of different techniques); on the contrary, the case $p > 2$ is not included in our treatment since the potential $f(t)$ becomes decreasing and degenerates at infinity; note that the use of a degenerate potential allows the approximation of the Mumford-Shah functional (the case $p > 4$).

Let us also point out that our theorem applies to the study of the singular perturbations of the rescaled Perona-Malik energy

$$\frac{1}{\varepsilon} \int_\Omega \log(1 + \varepsilon |\nabla u|^2) \, dx :$$

we will show that the right rescaling function is given by $r(\varepsilon) = \frac{\varepsilon}{\log \frac{1}{\varepsilon}}$ and that the family

$$\frac{1}{\varepsilon} \int_\Omega \log(1 + \varepsilon \nabla u)^2 \, dx + \left( \frac{\varepsilon}{\log \frac{1}{\varepsilon}} \right)^3 \int_\Omega \|\nabla^2 u\|^2 \, dx$$

$\Gamma$-converges to

$$\int_\Omega |\nabla u|^2 \, dx + c \int_\mathbb{S}_+ \sqrt{u^+ - u^-} \, d\mathcal{H}^1,$$

with $c > 0$ explicitly computable (see Example 4.8). The Perona-Malik functional was introduced in the context of Image Processing. Let us briefly recall the problem; if $g$ is the input grey level function representing the original image, the simplest way to smooth and denoise it is to apply a gaussian convolution kernel; this procedure turns out to be equivalent to letting $g$ evolve according to the heat equation, i.e. to taking as processed image the solution $u(x, t)$ of the heat diffusion equation

$$\frac{\partial}{\partial t} u = \Delta u \quad u(x, 0) = g(x), \quad (1.4)$$

computed at time $t$ ("$t$" can be seen as a scale parameter: the greater it is, the smaller is the scale at which the smoothing occurs).

The main drawback of this approach is that it produces an unconditional smoothing which cannot distinguish between objects and contours, since also edges begin soon to diffuse! To overcome these difficulties Perona and Malik proposed in [26] a model of selective smoothing where the contours are preserved as much as possible: it consists in replacing (1.4) by the nonlinear equation

$$\frac{\partial}{\partial t} u = \text{div} \left( \frac{\nabla u}{1 + |\nabla u|^2} \right) \quad u(x, 0) = g(x), \quad (1.5)$$

which is the gradient flow of the (Perona-Malik) functional $\int_\Omega \log(1 + |\nabla u|^2) \, dx$. The underlying idea is the following: where $|\nabla u|$ is large, in particular, near the edges, the diffusion is low and the contour is "kept", while far from the edges, where the gradient is small, $u$ diffuses as in the heat equation. Note that the simultaneous smoothing and edge detection effects of the equation strongly depend on the particular structure of the function $\log(1 + t^2)$: the quadratic behaviour near the origin is responsible of the denoising process while the concave and sublinear behaviour at infinity is responsible of the edge detection. Our $\Gamma$-convergence result says that there is an alternative procedure, based on minimizing the (rescaled) energy instead of considering its gradient flow, which exploits the structure of $\log(1 + t^2)$ generating again a smoothing and edge detection effect.

Actually, the same considerations apply to all functions $f$ satisfying our structure assumptions and we can think the functionals $\int f(|\nabla u|) \, dx$ as "generalized Perona-Malik energies" giving rise to "generalized Perona-Malik equations" of the form

$$\frac{\partial}{\partial t} u = \text{div} (g(\nabla u) \nabla u) \quad u(x, 0) = g(x), \quad (1.5)$$
with \(g\) bounded and decreasing to 0 when \(|\nabla u|\) is large.

We want to mention, as a further application of our main result, the study of the asymptotic behaviour of the family
\[
\frac{1}{\varepsilon} \int_{\Omega} f(\varepsilon|\nabla u|) \, dx + \varepsilon^3 \int_{\Omega} \|\nabla^2 u\|^2 \, dx,
\]
where \(f\) is non-decreasing, differentiable at the origin, with non-zero derivative, and concave at infinity: the \(\Gamma\)-limit turns out to be a functional defined in \(BV(\Omega)\) and taking the form
\[
f'(0) \int_{\Omega} |\nabla u| \, dx + \int_{S_u} \varphi(u^+ - u^-) \, d\mathcal{H}^{N-1} + f'(0) |D^\varepsilon u|,
\]
with \(\varphi\) explicitly characterized in terms of \(f\). Again, as \(f\) varies among all the admissible potentials, a wide class of jump-functions (satisfying (3.57)) can be generated (see Theorem 4.12 and Example 4.13).

Some final remarks are in order. All the convergence results we mentioned above are completely proved in the one-dimensional case; in \(N\) dimensions one can prove the following. Let \((F_n)\) be a sequence of one-dimensional functionals converging to \(F\) and denote by \((F_n^N)\) and \(F^N\) their respective \(N\)-dimensional versions; then we show that \(\Gamma\)-lim \(\lim_{n \to \infty} F^N_n(u) = F^N(u)\) if \(u\) satisfies
\[
\exists u_k \to u \quad \text{s.t.} \quad \mathcal{H}^{N-1}(S_{u_k}) < +\infty \quad \text{and} \quad F^N(u_k) \to F^N(u).
\]
The class of such functions coincides with the whole space if \(F^N\) is finite in \(BV\) so that in this case the \(H\)-convergence is completely proved; we believe that the same occurs when \(F^N\) is defined in \(SBV\) but, at the moment, such a technical result is not available, and, in fact, the representation of the \(\Gamma\)-limit is performed for functions with discontinuity set of finite \(\mathcal{H}^{N-1}\)-measure. Let us finally remark that these difficulties arise in the proof of the \(\Gamma\)-lim sup inequality; on the other hand, the \(\Gamma\)-lim inf inequality is completely proved as well as the equicoerciveness of the approximating functionals which guarantees the convergence of minimizers.

2 Preliminary results

2.1 Definitions and general properties of \(BV\) functions

In this subsection we fix notations and we briefly recall basic definitions and properties from the theory of \(BV\) functions; for a general treatment we refer to [6]. The Lebesgue measure and the \((N-1)\)-dimensional Hausdorff measure of a set \(B \subset \mathbb{R}^N\) are denoted by \(\mathcal{L}^N(B)\) and \(\mathcal{H}^{N-1}(B)\) respectively. We will often write \(|B|\) instead of \(\mathcal{L}^N(B)\). Given a measure \(\mu\) we denote its total variation by \(\mu\); moreover \(\mu|B\) denotes the restriction of the measure \(\mu\) to the set \(B\) given by \((\mu|B)(A) = \mu(B \cap A)\).

Let \(\Omega \subset \mathbb{R}^N\) be an open set, let \(u : \Omega \to \mathbb{R}\) be a measurable function, and let \(x \in \Omega\). We denote by \(u^+(x)\) and \(u^-(x)\), respectively, the upper and lower limit of \(u\) at \(x\), defined by
\[
\begin{align*}
\lim_{\rho \to 0^+} \frac{|\{y \in \Omega : |x-y| < \rho, \, u(y) > t\}|}{\rho^N} = 0
\end{align*}
\]
and
\[
\begin{align*}
\lim_{\rho \to 0^+} \frac{|\{y \in \Omega : |x-y| < \rho, \, u(y) < t\}|}{\rho^N} = 0.
\end{align*}
\]

If \(u^+(x) = u^- (x) \in \mathbb{R}\), then the common value of \(u^+(x)\) and \(u^-(x)\) is called the approximate limit of \(u\) at the point \(x\), and is denoted by \(\text{ap-lim}_{\rho \to 0^+} u(y)\).

We say that \(u\) is a \textit{function of bounded variation} in \(\Omega\), and we write \(u \in BV(\Omega)\), if \(u \in L^1(\Omega)\) and its distributional derivative is a vector-valued measure \(Du\) with finite total variation \(|Du|\). Given \(u \in BV(\Omega)\), we denote by by \(J_u\) the set where \(u^+ > u^-\) and by \(S_u\) the \textit{essential discontinuity set} of \(u\) made up of those points \(x\) which are not Lebesgue points. It turns out that \(J_u \subseteq S_u\) and \(\mathcal{H}^{N-1}(S_u \setminus J_u) = 0\). For every \(x \not\in S_u\) we denote by \(\tilde{u}(x)\) the approximate limit of \(u\) at \(x\).

The \textit{complete graph} of a function \(u \in BV(\Omega)\) is the set
\[
\Gamma_u := \{(x, z) \in \Omega \times \mathbb{R} : u^-(x) \leq z \leq u^+(x)\}.
\]
If \( u \in BV(\Omega) \), then it can be proved that \( S_u \) is countably \( (\mathcal{H}^N, 1, N - 1) \) rectifiable, i.e.

\[
S_u = N \cup \bigcup_{i \in \mathbb{N}} K_i,
\]

where \( \mathcal{H}^N(\Omega) = 0 \), and each \( K_i \) is a compact set contained in a \( C^1 \) hypersurface; as a consequence we have that for \( \mathcal{H}^N \)-a.e. \( x \in S_u \) it is possible to define an approximate tangent plane \( T_x(S_u) \) and therefore an approximate normal unit vector \( \nu_u(x) \) which can be chosen in such a way that

\[
\lim_{r \to 0^+} \int_{B_{r}^u(x)} |u(y) - u^+(x)| \, dy = 0,
\]

where \( B_{r}^u(x) := \{ y \in B_r(x) : \langle y - x, \nu_u(x) \rangle > 0 \} \) (here and in the sequel, given \( x \) and \( y \) in \( \mathbb{R}^N \), we denote the scalar product of \( x \) and \( y \) by \( x \cdot y \)). For every \( u \in BV(\Omega) \), by the Radon-Nykodim Theorem we can write \( Du = D^a u + D^s u \), where \( D^a u \) is absolutely continuous and \( D^s u \) is singular with respect to the Lebesgue measure. We denote the density of \( D^s u \) with respect to the Lebesgue measure by \( \nabla u \). Moreover, we denote the restriction of \( D^s u \) to \( S_u \) by \( D^s u^1 \), and the restriction of \( D^s u \) to \( \Omega \setminus S_u \) by \( D^s u^2 \). It turns out that \( D^s u = (u^1 + u^2) \nu_u \mathcal{H}^N \mid S_u \) so that in particular

\[
D^s u^1 = |D^s u| + (u^1 - u^2) \mathcal{H}^N \mid S_u.
\]

We will say that a set \( E \) is of finite perimeter in \( \Omega \) if \( \chi_E \) (i.e. the characteristic function of \( E \)) is of bounded variation in \( \Omega \). We define \( \partial^* E \cap \Omega := S_{\chi_E} \cap \Omega \) the reduced boundary of \( E \) in \( \Omega \). Let us recall now the Fleming-Rishel coarea formula. Let \( u \) be a Lipschitz function and let \( v \) belong to \( BV(\Omega) \). Then for almost every \( t \in \mathbb{R} \) we have that \( \{ x \in \Omega : u > t \} \) is a set of finite perimeter in \( \Omega \) and

\[
\int \nabla u \, v \, dx = \int_{-\infty}^{+\infty} \left( \int_{\partial^* \{ u > t \} \cap \Omega} \nu \, d\mathcal{H}^N \right) \, dt
\]

(2.1)

We say that \( u \) is a special function of bounded variation, and we write \( u \in SBV(\Omega) \), if \( u \in BV(\Omega) \) and \( D^s u = 0 \). For each \( p \geq 1 \) the space of all functions \( u \in SBV(\Omega) \) such that

\[
\nabla u \in L^p(\Omega) \quad \text{and} \quad \mathcal{H}^N \mid S_u < +\infty
\]

is denoted by \( SBV^p(\Omega) \). We consider also the larger space \( GBV(\Omega) \), which is composed by all measurable functions \( u : \Omega \to \mathbb{R} \) whose truncations \( u_k = (u \wedge k) \vee (-k) \) belong to \( BV(\Omega) \) for every \( k > 0 \); finally we set

\[
GSBV := \{ u \in GBV(\Omega) : |D^s u_k| = 0 \quad \forall k > 0 \} = \{ u \in L^1(\Omega) : u_k \in SBV(\Omega) \quad \forall k > 0 \},
\]

and

\[
GSBV^p(\Omega) := \{ u \in L^1(\Omega) : u_k \in SBV^p(\Omega) \quad \forall k > 0 \}.
\]

Every \( u \in GBV(\Omega) \cap L_{loc}^1(\Omega) \) has a countably \( (\mathcal{H}^N, 1, N - 1) \) rectifiable discontinuity set \( S_u \).

We conclude this subsection by recalling a "slicing" result due to Ambrosio (see [5]) and a \( L^1 \)-precompactness criterion by slicing proved in [1]. We introduce first some notation. Let \( \xi \in S^{N-1} \) and let \( \Pi_\xi := \{ y \in \mathbb{R}^N : y \cdot \xi = 0 \} \) be the linear hyperplane orthogonal to \( \xi \). Given \( E \subseteq \mathbb{R}^N \) we denote by \( E_\xi \subseteq \Pi_\xi \) the orthogonal projection of \( E \) on \( \Pi_\xi \) and for \( y \in \Pi_\xi \) we set \( E^\xi_y := \{ t \in \mathbb{R} : y + t\xi \in E \} \). Finally for \( u : E \to \mathbb{R} \) we define \( u^\xi_y : E^\xi_y \to \mathbb{R} \) by \( u^\xi_y(t) := u(y + t\xi) \).

**Theorem 2.1** a) Let \( u \in BV(\Omega) \). Then, for all \( \xi \in S^{N-1} \) the function \( u^\xi_y \) belongs to \( BV(\Omega^\xi_y) \) for \( \mathcal{H}^N \)-a.e. \( y \in \Pi_\xi \). For such \( y \) one has

\[
(u^\xi_y)'(t) = \nabla (y + t\xi) \cdot \xi \quad \text{for a.e. } t \in \Omega^\xi_y,
\]

\[
S_{u^\xi_y} = (S_u)^\xi_y,
\]

\[
u_{u^\xi_y} = (u^\xi_y(t)) = u^\xi(y + t\xi) \quad \text{or} \quad u^\xi_y(t) = u^\xi(y + t\xi),
\]

b) Let \( u \in BV(\Omega) \). Then, for all \( \xi \in S^{N-1} \) the function \( u^\xi_y \) belongs to \( BV(\Omega^\xi_y) \) for \( \mathcal{H}^N \)-a.e. \( y \in \Pi_\xi \). For such \( y \) we have

\[
\frac{d}{dt} u^\xi_y(t) = \frac{d}{dt} u^\xi_y(t), \quad \text{for a.e. } t \in \Omega^\xi_y
\]

\[
S_{u^\xi_y} = (S_u)^\xi_y,
\]

\[
u_{u^\xi_y} = (u^\xi_y(t)) = u^\xi(y + t\xi) \quad \text{or} \quad u^\xi_y(t) = u^\xi(y + t\xi),
\]
according to the case \( \nu_u \cdot \xi > 0 \) or \( \nu_u \cdot \xi < 0 \) (the case \( \nu_u \cdot \xi = 0 \) being negligible). Moreover we have
\[
\int_{\Pi_\xi} D^c u^y_\xi |(A_\xi^y) d\mathcal{H}^{N-1}(y) = |D^c u \cdot \xi|(A),
\]
for all open subset \( A \subseteq \Omega \), and for all Borel functions \( g \)
\[
\int_{\Pi_\xi} \sum_{t \in S_\xi} g(t) d\mathcal{H}^{N-1}(y) = \int_{S_u} g(x) |\nu_u \cdot \xi| d\mathcal{H}^{N-1}.
\]

b) Conversely, if \( u \in L^1(\Omega) \) and for all \( \xi \in \{e_1, \ldots, e_N\} \) and for a.e. \( y \in \Pi_\xi \) \( u^y_\xi \in BV(\Omega^y_\xi) \) \( (SBV(\Omega^y_\xi)) \) and
\[
\int_{\Pi_\xi} Du^y_\xi |d\mathcal{H}^{N-1}(y) < +\infty,
\]
then \( u \in BV(\Omega) \) \( (SBV(\Omega)) \).

Given a family \( \mathcal{F} \) of functions, for every \( \xi \in S^{N-1} \) and \( y \in \Pi_\xi \) we set \( \mathcal{F}^y_\xi := \{u^y_\xi : u \in \mathcal{F}\} \); moreover we say that a family \( \mathcal{F}' \) is \( \delta \)-close to \( \mathcal{F} \) if \( \mathcal{F}' \) is contained in a \( \delta \)-neighbourhood of \( \mathcal{F} \).

**Lemma 2.2** Let \( \mathcal{F} \) be a family of equiintegrable functions belonging to \( L^1(A) \) and assume that there exists a basis of unit vectors \( \{\xi_1, \ldots, \xi_N\} \) with the property that for every \( i = 1, \ldots, N \), for every \( \delta > 0 \), there exists a family \( \mathcal{F}_\delta \) \( \delta \)-close to \( \mathcal{F} \) such that \( (\mathcal{F}_\delta)^y_\xi \) is precompact in \( L^1(A^y_\xi) \) for \( H^{N-1} \)-a.e \( y \in A_\xi \). Then \( \mathcal{F} \) is precompact in \( L^1(A) \).

### 2.2 Semicontinuity and relaxation in \( BV \) and \( SBV \)

Let \( f : \mathbb{R} \rightarrow [0, +\infty] \) be convex. Then we define the recession function \( f^\infty \) of \( f \) by
\[
f^\infty(z) = \lim_{t \to +\infty} \frac{f(tz)}{t}.
\]
Let \( \theta : \mathbb{R} \rightarrow [0, +\infty] \) be lower semicontinuous and such that there exists \( \lim_{t \to +\infty} \theta(t)/t \). Then we can define the recession function \( \theta^0 \) of \( \theta \) by
\[
\theta^0(z) = \lim_{t \to +\infty} \frac{\theta(tz)}{t}.
\]
The functions \( f^\infty \) and \( \theta^0 \) turn out to be 1-homogeneous. For every \( g, h : \mathbb{R} \rightarrow [0, +\infty] \), we define the inf-convolution of \( g \) and \( h \) as the function \( g \triangle h \) given by
\[
(g \triangle h)(z) = \inf \{g(x) + h(z - x) : x \in \mathbb{R}\}.
\]
Finally we recall that given a function \( F : X \rightarrow \mathbb{R} \cup +\infty \), where \( X \) is a topological space, we denote by \( F^0 \) the relaxed functional of \( F \), i.e. the greatest lower semicontinuous (with respect to the \( X \)-topology) functional which is less than \( F \).

The following relaxation result is proved in [8].

**Theorem 2.3** (Relaxation in \( BV \)) Let \( f : [0, +\infty) \rightarrow [0, +\infty) \) be a non-decreasing convex function and let \( \varphi : [0, +\infty) \rightarrow [0, +\infty) \) be a concave function. Let \( F : BV(\Omega) \rightarrow [0, +\infty] \) be defined by
\[
F(u) := \begin{cases} 
\int_{\Omega} f(\nabla u) dx + \int_{S_u} \varphi(u^+ - u^-) d\mathcal{H}^{N-1} & \text{if } u \in SBV^2(\Omega) \cap L^\infty(\Omega), \\
+\infty & \text{otherwise}.
\end{cases} \tag{2.2}
\]
Then the relaxed functional of \( F \) with respect to the \( L^1 \)-metric is given on \( BV \) by
\[
F^0(u) := \int_{\Omega} f_1(|\nabla u|) dx + \int_{S_u} \varphi_1(u^+ - u^-) d\mathcal{H}^{N-1} + (f^\infty(1) \wedge \varphi^0(1)) |D^c u|,
\]
where \( f_1 := f \triangle \varphi^0 \) and \( \varphi_1 := \varphi \triangle f^\infty \).
It is possible to prove that \( f \triangle \varphi^D = [f \wedge (\varphi^D + f(0))]^{**} \), where \( h^{**} \) denotes the convexification of \( h \), i.e. the greatest convex and lower semicontinuous function which is smaller than \( h \) and, analogously, \( \varphi \triangle f^\infty = \text{sub} [\varphi \wedge (f^\infty + \varphi(0))] \) where \( \text{sub} h \) denotes the subadditive envelope, i.e. the greatest lower semicontinuous and subadditive function which is smaller than \( h \). Given two Borel functions \( \varphi : [0, +\infty] \to [0, +\infty) \) and \( f : [0, +\infty] \to [0, +\infty) \), we consider the functional \( F \) defined by

\[
F(u) = \begin{cases} 
\int_{\Omega} f(|\nabla u|) \, dx + \int_{S_u} \varphi(u^+ - u) \, dH^N & \text{if } u \in GSBV(\Omega), \\
\infty & \text{otherwise.}
\end{cases}
\tag{2.3}
\]

In [5] the following semicontinuity result is proved.

**Theorem 2.4 (Ambrosio’s Semicontinuity Theorem)** Let \( \Omega \subset \mathbb{R}^N \) be an open bounded set. Let \( f : [0, +\infty) \to [0, +\infty) \) be a non-decreasing convex function such that \( f^\infty(1) = +\infty \) and let \( \varphi : [0, +\infty) \to [0, +\infty) \) be a non-decreasing subadditive function such that \( \varphi^D(1) = \infty \). Then the functional \( F \) defined in (2.3) is lower semicontinuous with respect to the \( L^1 \) convergence.

### 2.3 A density result in \( SBV \)

In analogy with the strong density results of smooth functions in \( W^{1,p}(\Omega) \), functions in \( SBV^p(\Omega) \) can be approximated in a “strong sense” by functions which have a “regular” jump set and are smooth outside. This can be formally expressed as follows.

Let \( \Omega \) be an open bounded subset in \( \mathbb{R}^N \) with Lipschitz boundary and denote by \( \mathcal{W}(\Omega) \) the space of all function \( w \in SBV(\Omega) \) enjoying the following properties:

i) \( \mathcal{H}^N(S_w \setminus S_w) = 0 \);

ii) \( S_w \) is the intersection of \( \Omega \) with the union of a finite number of pairwise disjoint \( \mathcal{N} \) -simplexes;

iii) \( w \in W^{k,\infty}(\Omega \setminus S_w) \) for every \( k \in \mathbb{N} \).

Cortesani and Toader have proved in [18] the following density result.

**Theorem 2.5** Let \( u \in SBV^p(\Omega) \cap L^\infty(\Omega) \). Then there exists a sequence \( (w_j)_j \) in \( \mathcal{W}(\Omega) \) such that \( w_j \to u \) strongly in \( L^1(\Omega) \), \( \nabla w_j \to \nabla u \) strongly in \( L^p(\Omega, \mathbb{R}^N) \), \( \lim_j \|w_j\|_\infty = \|u\|_\infty \) and

\[
\limsup_{j \to \infty} \int_{S_{w_j}} \phi(w_j^+, w_j, \nu_{w_j}) \, dH^N 1 \leq \int_{S_u} \phi(u^+, u, \nu_u) \, dH^N 1,
\]

for every upper semicontinuous function \( \phi : \mathbb{R} \times \mathbb{R} \times S^N \to [0, +\infty) \) such that \( \phi(a, b, \nu) = \phi(b, a, -\nu) \), for every \( a, b \in \mathbb{R} \) and for every \( \nu \in S^N \).

### 2.4 \( \Gamma \)-convergence

We recall here the definition and the main properties of \( \Gamma \)-convergence: for the general theory we refer to [19] (see also the forthcoming [11]).

**Definition 2.6** Let \( (X, d) \) be a metric space and let \( F_h : X \to \mathbb{R} \cup \{+\infty\} \) be a sequence of functions. We set

\[
\Gamma \text{-lim inf} \ F_h(x) := \inf \left\{ \liminf_{h \to \infty} F_h(x_h) : x_h \to x \right\}
\]

and

\[
\Gamma \text{-lim sup} \ F_h(x) := \inf \left\{ \limsup_{h \to \infty} F_h(x_h) : x_h \to x \right\}.
\]
We say that the sequence \((F_h)_{h \in \mathbb{N}}\) \(\Gamma\)-converges if
\[
\Gamma \liminf_{h \to \infty} F_h(x) = \Gamma \limsup_{h \to \infty} F_h(x) \quad \forall x \in X.
\]
The common value is called \(\Gamma\)-limit and is denoted by \(\Gamma \lim_{h \to \infty} F_h\).

**Definition 2.7** We say that the maps \(F_h : X \to \mathbb{R} \cup \{+\infty\}\) are equicoercive if for every \(t \in \mathbb{R}\) there exists a compact subset \(K_t \subseteq X\) such that
\[
\{x \in X : F_h(x) \leq t\} \subseteq K_t \quad \forall h \in \mathbb{N}.
\]

The following theorem explains the variational meaning of this kind of convergence.

**Theorem 2.8** Let \((F_h)_{h}\) be a sequence of equicoercive maps which \(\Gamma\)-converges to \(F\). Then, if \((x_h)_{h}\) is a sequence such that
\[
\lim_{h \to \infty} F_h(x_h) = \lim \inf_{h \to \infty} F_h,
\]
then \(x_h\) is precompact and any cluster point is a minimizer of \(F\).

We finally recall that given \(F : X \to \mathbb{R} \cup \{+\infty\}\), the relaxed functional \(\overline{F}\) can be characterized as the \(\Gamma\)-limit of the constant sequence \(F_n = F\) for every \(n \in \mathbb{N}\).

3 The main convergence result in the one-dimensional case

Let \(f_n : [0, +\infty) \to [0, +\infty)\) be a family of continuous non-decreasing functions and let \(r_n\) be an infinitesimal sequence of positive real numbers. For any open bounded subset \(I \subseteq \mathbb{R}\), we define

\[
F_n(u) := \begin{cases} 
\int_I f_n(|u'|) \, dx + (r_n)^{3/2} \int_I |u''|^2 \, dx & \text{if } u \in W^{2,2}(I), \\
+ \infty & \text{otherwise in } L^1(I).
\end{cases}
\] (3.1)

Moreover, given two functions \(b, g : [0, +\infty) \to [0, +\infty)\), we set

\[
\mathcal{F}_{b,g}(u) := \begin{cases} 
\int_I g(|u'|) \, dx + \sum_{s_u} \varphi(u^+ - u) & \text{if } u \in SBV(I), \\
+ \infty & \text{otherwise in } L^1(I),
\end{cases}
\] (3.2)

where
\[
\varphi(z) := \inf_{\eta > 0} \left\{ \int_0^\eta b(|u'|) \, dx + \int_0^\eta u''^2 \, dx : u \in W^{2,2}(0, \eta), u(0) = 0, u(\eta) = z, u'(0) = u'(\eta) = 0 \right\}.
\] (3.3)

If \(g\) is convex and \(b\) is convex or concave or convex-concave then we can finally define

\[
F_{b,g}(u) := \begin{cases} 
\int_I g_1(|u'|) \, dx + \sum_{s_u} \varphi_1(u^+ - u) + (g^\infty(1) \wedge b^\infty(1)) |D^c u| & \text{if } u \in BV(I), \\
+ \infty & \text{otherwise in } L^1(I),
\end{cases}
\] (3.4)

where \(g_1 := g \triangle b^\infty = [g \wedge (b^\infty + g(0))]^*\) and \(\varphi_1 := \varphi \triangle g^\infty = \text{sub} (\varphi \wedge g^\infty)\) \((g^\infty\) and \(b^\infty\) are the recession functions of \(g\) and \(b\) respectively defined in Subsection 2.2).

**Remark 3.1** Note that if \(g^\infty(1) = b^\infty(1) = +\infty\) then \(F_{b,g} = \mathcal{F}_{b,g}(u)\).
Our main result is stated in the following theorem.

**Theorem 3.2** Let \( f_n \) and \( r_n \) be as above and satisfying in addition the following hypotheses:

i) there exists a non-decreasing function \( g : [0, +\infty) \to [0, +\infty) \) such that
\[
f_n(t) \to g(t) \quad \forall t \in [0, +\infty);
\]  
(3.5)

ii) there exists a non-decreasing and continuous function \( b : (0, +\infty) \to (0, +\infty) \) such that
\[
r_n f_n \left( \frac{t}{r_n} \right) \to b(t) \quad \forall t > 0.
\]  
(3.6)

Then
\[
\Gamma \limsup_{n \to \infty} F_n \leq F_{b,g}(u),
\]
with respect to the \( L^1(I) \)-convergence, where \( F_n \) are the functionals defined in (3.1) while \( F_{b,g} \) denotes the \( L^1 \)-relaxation of the functional \( F_{b,g} \) introduced in (3.2). If in addition we assume

iii) one of the two following structure conditions holds true:

**st1**) \( f_n \) is convex for every \( n \in \mathbb{N} \);

**st2**) there exists a sequence \( \{x_n\} \subset (0, +\infty) \) such that \( x_n \to +\infty \) and \( f_n \) is convex in \([0, x_n]\) and concave in \([x_n, +\infty)\),

then
\[
\Gamma \lim_{n \to \infty} F_n = F_{b,g} = F_{b,g},
\]
where \( F_{b,g} \) is the functional defined in (3.4). Finally, every sequence \( u_n \) such that \( \sup_n (F_n(u_n) + \|u_n\|_1) < +\infty \)
is strongly precompact in \( L^p \) for every \( p \geq 1 \).

**Remark 3.3** If iii) holds then \( g \) is convex; concerning \( b \), assumption **st1** implies that it is in turn convex while **st2** implies that it is either concave or convex-concave. In all these cases the recession function \( b^0 \) is well defined. We finally point out that the equality \( F_{g,b} = F_{b,g} \) stated in the last part of the theorem is a consequence of Theorem 2.3 and of the equality \( \varphi^0 = b^0 \) which will be proved in the sequel (see Lemma 3.6).

**Remark 3.4** If **st2**) holds with \( \limsup_{n \to \infty} x_n r_n = c > 0 \), then
\[
b(t) \leq g^\infty(t) \quad \forall t \in [0, c],
\]  
(3.7)
so that, in particular, \( b^0(1) \geq g^\infty(1) \). Passing to a subsequence, if needed, we can suppose that \( \lim_n x_n r_n = c \); since the functions \( f_n \) pointwise converge to \( g \) and becomes convex in larger and larger intervals, we have
\[
g'(t- \leq \liminf_{n \to \infty} f'_n(t- \leq \limsup_{n \to \infty} f'_n(t+ \leq g'(t+),
\]  
(3.8)
for every \( t > 0 \). Suppose that \( g^\infty(1) \neq 0 \), otherwise the statement is trivial and let \( y_k \to +\infty \) such that \( g'(y_k) \to g^\infty(1) \). For a fixed \( k \) and \( \delta \in (0, 1) \) there exists \( n_{k,\delta} \) such that \( x_n \geq y_k \), \( f_n(t) \geq g(y_k)/2 \), and \( f'_n(t) \geq (1-\delta)g'(y_k) \) for every \( n \geq n_{k,\delta} \), so that, by convexity,
\[
f_n(t) \geq \frac{g(y_k)}{2} + (1-\delta)g'(y_k)(t-y_k) \quad \forall t \in [y_k, x_n] \forall n \geq n_{k,\delta}.
\]  
(3.9)
Fix \( t < c \); then, by (3.9),
\[
r_n f_n \left( \frac{t}{r_n} \right) \geq r_n g(y_k)/2 + r_n (1-\delta)g'(y_k) \left( \frac{t}{r_n} - y_k \right),
\]
for every \( n \geq \pi \), where \( \pi \geq n_{k,\delta} \) is such that \( y_k \leq t/r_n \leq x_n \) for every \( n \geq \pi \). Passing to the limit in \( n \) in the above inequality and taking into account (3.6) we obtain \( b(t) \geq (1-\delta)g'(y_k)t \), from which (3.7) follows letting \( k \) tend to infinity and then \( \delta \) tend to zero. With the same proof we see that **st1** implies that \( b(t) \geq g^\infty(t) \) for every \( t \geq 0 \).
Before giving the proof of the theorem we need to state and prove some preparatory lemmas.

**Lemma 3.5** Suppose that \( b(t) = Mt \) for some \( M > 0 \) and let \( \varphi \) be the function defined in (3.3). Then \( \varphi(z) = Mz \) for every \( z > 0 \).

**Proof.** Fix \( z > 0 \) and let \((v, \eta)\) be an admissible pair for problem (3.3), then

\[
\int_0^T M |v'|^2 \, dt + \int_0^T |v''|^2 \, dt \geq \int_0^T M |v'|^2 \, dt \geq Mz,
\]

and therefore \( \varphi(z) \geq Mz \). Let us prove now the reverse inequality. To this aim we construct a sequence of admissible pairs \((v_n, \eta_n)\) by setting \( \eta_n := nz \) and

\[
v_n(t) := \begin{cases}
\frac{\phi(t)}{n} & \text{if } t \in [0, 1) \\
\frac{1}{n} + \frac{1}{n}(t - 1) & \text{if } t \in [1, nz - 1) \\
z - \frac{\phi(nz, t)}{n} & \text{if } t \in [nz - 1, nz],
\end{cases}
\]

where \( \phi \) is a function belonging to \( C^2([0, 1]) \) and satisfying \( \phi(0) = \phi'(0) = 0 \), \( \phi(1) = \phi'(1) = 1 \). We can now estimate

\[
\varphi(z) \leq \int_0^T M |v'|^2 \, dt + \int_0^T |v''|^2 \, dt = 2 \int_0^1 |\varphi'|^2 \, dt + M^{n} z^2 - 2 + 2 \int_0^1 |\varphi''|^2 \, dt
\]

and therefore, letting \( n \to \infty \), we obtain \( \varphi(z) \leq Mz \).

\( \square \)

**Lemma 3.6** Let \( b \) as in Remark 3.3. Then the function \( \varphi : [0, +\infty) \to [0, +\infty) \) defined in (3.3) is continuous, non-decreasing, subadditive, and \( \varphi^0(1) = b^0(1) \).

**Proof.** The first three properties are easy; let us prove only the last one. We begin with the case

\[
b^0(1) = +\infty.
\]

**Claim.** For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) with the following property: if \( z < \delta \) and if \((\eta, u)\) is an admissible pair for problem (3.3) satisfying

\[
\int_0^\eta b(|u'|) \, dx + \int_0^\eta |u''|^2 \, dx < (1 + \varepsilon)\varphi(z),
\]

then \( |u'| \leq \varepsilon \) in \((0, \eta)\).

Suppose by contradiction the existence of \( \varepsilon > 0 \) and of a sequence \( \delta_n \downarrow 0 \) such that, for every \( n \in \mathbb{N} \), there exist \( z_n < \delta_n \) and \((\eta_n, u_n)\) which satisfies

\[
\int_0^{\eta_n} b(|u'|) \, dx + \int_0^{\eta_n} |u''|^2 \, dx < (1 + \varepsilon)\varphi(z_n),
\]

and

\[
||u_n||_{L^\infty(0, \eta_n)} > \varepsilon.
\]

Note that we can suppose \( \eta_n > 1 \) for every \( n \) (if needed \( u_n \) can be extended outside the original interval as the constant function \( z_n \)); using Hölder’s inequality we can estimate, for every \( x, y \in (0, \eta_n) \)

\[
|u'_n(x) - u'_n(y)| \leq \int_x^y |u''_n| \, dt \leq \sqrt{|x - y|} \left( \int_0^{\eta_n} |u''_n|^2 \, dt \right)^{1/2} \leq C \sqrt{|x - y|},
\]
where \( C > 0 \) is independent of \( n \); by the above estimate and by (3.13) we can state the existence of an interval \( I_n \subseteq (0, \eta_n) \) such that \( |I_n| \geq C' \), with \( C' \) independent of \( n \), and \( |u_n'| \geq \frac{\varepsilon}{2} \) in \( I_n \). As a consequence we deduce
\[
\int_0^{\eta} b(u'_n) \, dx + \int_0^{\eta} |u''_n|^2 \, dx \geq \int_{I_n} b(|u'_n|) \, dx \geq b \left( \frac{\varepsilon}{2} \right) C'
\]
which is in contradiction with (3.12) since \( \varphi(z_n) \), by continuity, tends to 0. The claim is proved. Given \( M > 0 \), thanks to (3.10), we can choose \( \varepsilon \) such that \( b(t)/t \geq M \) for every \( t \in (0, \varepsilon] \); if \( \delta > 0 \) is as in the above Claim, for \( 0 < z < \delta \) we can estimate
\[
(1 + \varepsilon)\varphi(z) \geq \int_0^{\eta} b(u'_n) \, dx + \int_0^{\eta} |u''_n|^2 \, dx \geq \int_0^{\eta} \frac{b(u'_n)}{|u'_n|} \, dx \geq M \int_0^{\eta} |u'| \, dx \geq Mz,
\]
where \( (\eta, u) \) is an admissible pair satisfying (3.11); this concludes the proof when (3.10) holds. Let us suppose now that
\[
b^0(1) = C < +\infty. \tag{3.14}
\]
Fix \( \sigma > 0 \) and choose \( \varepsilon \sigma > 0 \) such that \( b(t) < (C + \sigma)t \) for any \( t \in (0, \varepsilon \sigma) \). Consider the sequence of admissible pairs \( (\eta_n, v_n) \) constructed in the previous lemma; for \( n \) large we have \( \|u'_n\|_{\infty} \leq \varepsilon \sigma \) and therefore
\[
\varphi(z) \leq \int_0^{\eta_n} b(v'_n) \, dt + \int_0^{\eta_n} v''_n \, dt \leq (C + \sigma) \int_0^{\eta_n} |v'_n| \, dt + \int_0^{\eta_n} |v''_n|^2 \, dt = (C + \sigma)z + O \left( \frac{1}{n} \right).
\]
Letting \( n \to \infty \) and \( \sigma \to 0 \) we obtain
\[
\varphi(z) \leq Cz \quad \forall z > 0. \tag{3.15}
\]
Finally, arguing exactly as for the other case, we easily obtain \( \liminf_{z \to 0^+} \varphi(z)/z \geq C \), which concludes the proof of the lemma. \( \square \)

**Lemma 3.7** Let \( (u_n)_{n \in \mathbb{N}} \) be a sequence of functions such that \( \sup_n F_n(u_n) < +\infty \) and, for a fixed \( c > 0 \), consider the sets \( D_n := \{ x \in I : |u'_n(x)| > c/r_n \} \). Then there exists \( \bar{n} \in \mathbb{N} \), depending on \( c \), such that
\[
|D_n| \leq \left( \frac{2 \sup_n F_n(u_n)}{b(c)} \right) r_n,
\]
for every integer \( n \geq \bar{n} \).

**Proof.** We can estimate
\[
F_n(u_n) \geq \int_{D_n} f_n(u'_n) \, dx \geq \frac{r_n}{r_n} f_n \left( \frac{c}{r_n} \right) |D_n| \geq \frac{1}{2r_n b(c)} |D_n|,
\]
if \( n \) is large enough, thanks to (3.6). \( \square \)

**Lemma 3.8** Suppose that also iii) of Theorem 3.2 holds true and let \( (u_n) \) be such that
\[
\sup_n F_n(u_n) < +\infty.
\]
Then
\[ r_n \| u_n' \|_\infty \leq 2 \sup_n F_n(u_n) + 1, \]  
(3.16)
for \( n \) large enough. Moreover, if \( g \neq 0 \), there exists a positive constant \( C \) depending only on \( I \), \( g \), and \( b \) such that
\[ \text{Var} u_n \leq C \left( \sup_n F_n(u_n) + 1 \right)^2, \]  
(3.17)
for \( n \) large enough.

**Proof.** Take \( c = 1 \) and consider the sets \( D_n \) defined in the previous lemma; since they are open, we can write \( D_n = \bigcup_{m=1}^{\infty} (a_m^k, b_m^k) \). Let \( y \) be a point of \( D_n \); therefore there exists \( k \in \mathbb{N} \) such that \( y \in (a_m^k, b_m^k) \). By Lemma (3.7) and using Hölder’s Inequality, we have
\[
|u_n'(y)| \leq |u_n'(a_m^k)| + \int_{a_m^k}^{y} |u_n''(t)| \, dt 
\leq \frac{1}{r_n} + \frac{|D_n|^\frac{1}{2}}{(r_n)^3} \left( \sup_n \left( \frac{1}{r_n} \int_{a_m^k}^{b_m^k} |u_n''|^2 \, dt \right) \right)^\frac{1}{2} 
\leq \frac{1}{r_n} + \frac{2(r_n)^{\frac{1}{2}}}{(r_n)^3} \sup_n F_n(u_n) = \left( 2 \sup_n F_n(u_n) + 1 \right) \frac{1}{r_n},
\]
so that (3.16) is proved. Concerning the second part of the Lemma, we first observe that, by a translation argument, we can suppose that \( f_n(0) = g(0) = 0 \) for every \( n \in \mathbb{N} \). Let \( x_0 \) be the last point such that \( g(x_0) = 0 \) and define
\[
\tilde{g}(x) := \begin{cases} 0 & \text{if } x \in [0, x_0], \\ g(x - x_0) & \text{if } x \geq x_0; \end{cases}
\]
it is easy to see that \( \tilde{g} \) is still convex, \( \tilde{g}^\infty = g^\infty \), and, taking into account the fact that \( f_n \rightarrow g \) uniformly on compact subsets of \([0, +\infty)\) (the uniformity follows from the pointwise convergence and from the monotonicity of \( f_n \)),
\[ \forall \delta \in (0, 1), \forall K > 0, \exists \pi \text{ s.t. } f_n \geq (1 - \delta)\tilde{g} \text{ in } [0, K], \forall n \geq \pi. \]  
(3.18)
Fix \( \bar{g} > 0 \) such that \( \bar{g}'(\bar{g}) > \bar{g}^\infty(1/2); \) set \( k := 2 \sup_n F_n(u_n) + 1 \) and let \( \bar{x} \) be the first point such that \( \bar{g}'(\bar{x} +)/2 \geq \min\{g^\infty(1)/3, b(1)/(3k)\} \). Since either \( \bar{x} = 0 \) or
\[
\frac{\bar{g}'(\bar{x} -)}{2} \leq \min \left\{ \frac{g^\infty(1)}{3}, \frac{b(1)}{3k} \right\} \leq \frac{\bar{g}'(\bar{x} +)}{2}, \]  
(3.19)
it is clear that \( \bar{x} < \bar{g} \). So, by virtue of (3.18), (3.19), (3.8), and (3.6), we can find \( \pi \) such that
\begin{enumerate}
  \item[a)] \( f_n \geq \tilde{g}(x)/2 \text{ in } [0, \bar{x} + 1], \)
  \item[b)] \( f_n'(\bar{x} +) \geq \min\{g^\infty(1)/3, b(1)/(3k)\}, \)
  \item[c)] \( f_n(k/r_n)/(k/r_n) \geq b(k)/(3k), \)
\end{enumerate}
for every \( n \geq \pi \); we define \( a(t) := \tilde{g}(\bar{x})/2 + \min\{g^\infty(1)/3, b(1)/(3k)\}(t - \bar{x}) \). Exploiting the convexity of \( \tilde{g} \), we observe that, by (3.19) and a),
\[ a(t) \leq \frac{\tilde{g}(\bar{x})}{2} \leq f_n(t) \text{ in } [0, \bar{x} + 1]. \]  
(3.20)
If \textbf{st1)} holds, that is if \( f_n \) is convex, then, taking into account b) and (3.20), we also have
\[
a(t) \leq f_n(t) + f_n'(t)(t-x) \leq f_n(t) \quad \forall t \geq x + 1.
\]
Suppose now that \textbf{st2)} holds; by replacing \( f_n \) with
\[
\tilde{f}_n(t) := \begin{cases} f_n(t) + (r_n t)^2 & \text{if } t \leq x_n, \\ f_n(t) + (r_n x_n)^2 & \text{if } t > x_n, \end{cases}
\]
if needed, we can assume that \( f_n \) is strictly convex in \([0, x_n]\) (recall that \( x_n \) is the point appearing in condition \textbf{st2)}). Arguing as above, we obtain
\[
a(t) \leq f_n(t) \quad \forall t \in [x + 1, x_n \wedge k/r_n], \tag{3.21}
\]
Let us denote by \( y_n \) the first strictly positive point such that \( f_n(y_n) = [f_n(k/r_n)/(k/r_n)]y_n \); by the strict convexity assumption we have that \( 0 < y_n \leq k/r_n \). If \( x_n < y_n < k/r_n \), we can first observe that, by concavity,
\[
f_n'(t \pm) \geq f_n'(y_n -) \geq \frac{r_n}{k} f_n \left( \frac{k}{r_n} \right) \quad \forall t \in (x_n, y_n), \tag{3.22}
\]
where the last inequality is a consequence of the following one
\[
f_n(t) \geq f_n(y_n) + \frac{r_n}{k} f_n \left( \frac{k}{r_n} \right) (t - y_n) = \frac{r_n}{k} f_n \left( \frac{k}{r_n} \right) t \quad \text{in } [y_n, k/r_n],
\]
where we used again the concavity of \( f_n \) in \((x_n, y_n)\). Using (3.22) and c) we then have
\[
a(t) \leq f_n(x_n) + \frac{r_n}{k} f_n \left( \frac{k}{r_n} \right) (t - x_n) \leq f_n(t) \quad \text{in } [x_n, y_n] \tag{3.23}
\]
and therefore
\[
a(t) \leq f_n(y_n) + \frac{r_n}{k} f_n \left( \frac{k}{r_n} \right) (t - y_n) \leq f_n(t) \quad \text{in } [y_n, k/r_n].
\]
If \( x_n < y_n = k/r_n \), then either
\[
f_n(t) > \frac{r_n}{k} f_n \left( \frac{k}{r_n} \right) t \geq \frac{b(k)}{3k} t \quad \text{in } [0, k/r_n],
\]
or
\[
f_n(t) < \frac{r_n}{k} f_n \left( \frac{k}{r_n} \right) t.
\]
In the first case, (3.17) follows immediately; in the second case, we observe that (3.22) and therefore (3.23) are still true. Summarizing, we have proved that
\[
a(t) \leq f_n(t) \quad \text{in } [0, k/r_n], \tag{3.24}
\]
if \( x_n < y_n \); arguing in a similar way, we obtain the same estimate also if \( y_n \leq x_n \). Using the definition of \( a(t) \) and the fact that \( x < y \), from (3.24) we easily obtain
\[
\min \left\{ \frac{g^\infty(1)}{3}, \frac{b(1)}{3} \right\} t \leq \frac{k g^\infty(1)}{3} y \quad \forall t \in [0, k/r_n],
\]
from which, recalling the definition of \( k \) and (3.16), the inequality (3.17) follows with
\[
C := (6 + 2g^\infty(1)/3)(k g^\infty(1)/3) \cdot (\min \{g^\infty(1)/3, b(1)/3\}) \leq 1.
\]
\[\square\]
**Remark 3.9** Let us remark that if \( u_n \to u \) in \( L^1 \) and \( \sup_n F_n(u_n) < +\infty \) then \( u_n \to u \) in \( L^p \) for every \( p \geq 1 \): indeed from (3.17) it easily follows that \( u_n \) is equibounded in \( L^\infty \). As a consequence we have that in one dimension the functionals \( F_n \) \( \Gamma \)-converge with respect to the \( L^1 \)-norm if and only if they \( \Gamma \)-converge with respect to the \( L^p \)-norm, for every \( p \geq 1 \).

**Lemma 3.10** Assume that also condition iii) of Theorem 3.2 holds and let \((u_n)_{n \in \mathbb{N}} \subset SBV(I)\) be such that \( r_n \|u'_n\|_{\infty} \to 0 \) as \( n \to \infty \). Then there exists an increasing sequence \((\psi_i)_{i \in \mathbb{N}}\) of positive convex functions enjoying the following properties:

i) \( \psi_i(t) \uparrow g_i \) for every \( t > 0 \) as \( i \to \infty \) (we recall that \( g_1 \) is the function appearing in (3.4));

ii) \( \psi_i^\infty(1) = g_i^\infty(1) = b^0(1) \wedge g^\infty(1) \) for every \( i \);

iii) passing to a subsequence, still denoted by \((u_n)_n\), we have that for every \( i \) there exists \( n_i \) such that

\[
f_n(|u'_n|) \geq \psi_i(|u'_n|),
\]

for every \( n \geq n_i \).

**Proof.** We can assume that \( \min\{g^\infty(1), b^0(1)\} \neq 0 \), otherwise the statement is trivial. We will distinguish two cases.

**Case 1:** \( g^\infty(1) > b^0(1) \).

Note that in this case, by Remark 3.4, we have that necessarily st2 holds true, with \( \lim_n x_n r_n = 0 \). We can suppose without loss of generality that \( f_n(0) = g(0) = 0 \) for every \( n \in \mathbb{N} \) (otherwise translate). We begin by assuming

\[
g'(0+) < b^0(1),
\]

so that, letting \( \overline{x} \) be the last point such that \( g'(\overline{x} - ) \leq b^0(1) \leq g'(\overline{x} +) \) and setting \( \overline{y} := \sup\{y \geq 0 : g(t) \leq b^0(1)t, \forall t \in [0, y]\} \), we have \( \overline{x} < \overline{y} \). We make also the following assumption:

\[
\forall \delta \in (0, 1), \forall K > 0, \exists n_{\delta, K} \text{ s.t } f_n(t) \geq (1 - \delta)g(t) \quad \forall t \in [0, K] \text{ and } \forall n \geq n_{\delta, K}.
\]

(3.26)

It is clear that we can find \( \delta_0 \in (0, 1) \) such that for every \( 0 < \delta \leq \delta_0 \) there holds \( (1 - \delta)g^\infty(1) > b^0(1) \) and \( \overline{x}_0 < \overline{y} \), where \( \overline{x}_0 \) is the last point such that \( (1 - \delta)g'(\overline{x}_0 - ) \leq b^0(1) \leq (1 - \delta)g'(\overline{x}_0 +) \). In particular, for every \( \delta \leq \delta_0 \), there exists \( x_\delta \in [\overline{x}_0, \overline{y}] \) satisfying

\[
(1 - \delta)g'(x_\delta) \geq b^0(1).
\]

(3.27)

Let us choose now a sequence \( d_n \) increasing to \(+\infty\) with the following properties:

a) \( d_n > \|u'_n\|_{\infty} \) and \( d_n > x_n \) for every \( n \in \mathbb{N} \), where \( x_n \) is the point appearing in assumption st2;

b) \( d_n r_n \to 0 \) so slowly that \( r_n f_n(d_n)/b(d_n r_n) \to 1 \) (this is possible thanks to (3.6)).

Setting \( \overline{z}_n := f_n(d_n)/d_n \), from b) it easily follows that \( \lim_{n \to \infty} \overline{z}_n = b^0(1) \), so that, passing to a subsequence if needed and denoting \( s_n := \overline{z}_n \wedge b^0(1) \), we have that \( s_n \) is a non-decreasing sequence converging to \( b^0(1) \). Finally, denoting by \( y_n \) the first strictly positive point such that \( f_n(y_n) = s_n y_n \), the convergence of \( s_n \) to \( b^0(1) \) and of \( f_n \) to \( g \) implies \( y_n \to \overline{y} \). Taking into account all this facts and recalling (3.8), it is now evident that we can find \( \overline{z}_0 > 0 \) such that

*) \( f_n(t) \geq (1 - \delta)g(t) \), for every \( t \in [0, x_\delta] \),

**) \( f_n'(x_\delta-) \geq b^0(1) \geq s_n \) and \( x_\delta < y_n \).
for every $n \geq n_\delta$. At this point, for $k > n_\delta$ we define the function $\psi^k_\delta$ by induction in the following way:

$$
\psi^k_\delta = [(1 - \delta)g \wedge s_k]^* \text{ in } [0,d_k] \quad \text{and} \quad \psi^k_\delta = \psi^k_\delta( t_j) + s_{j+1}( t - t_j) \text{ in } [d_j,d_{j+1}], \text{ for } j \geq k.
$$

Recalling that $s_n$ increases to $b^0(1)$ it is easily seen that $\psi^k_\delta$ is convex with $(\psi^k_\delta)^* (1) = b^0(1)$ and $\psi^k_\delta \uparrow [(1 - \delta)g \wedge b^0(1)t]^*$ as $k$ tends to infinity. Defining

$$
\tilde{f}_n(t) := \begin{cases} f_n(t) & \text{if } t \in [0,x_\delta]; \\
 f_n(x_\delta) + s_n(t-x_\delta) & \text{otherwise},
\end{cases}
$$

by *) and **), we have

$$
\psi^k_\delta(t) \leq \tilde{f}_n(t) \quad \text{in } [0,d_n]. \quad (3.28)
$$

Moreover it turns out

$$
\tilde{f}_n(t) \leq f_n(t) \quad \text{in } [0,y_n]; \quad (3.29)
$$

actually, this is true in $[0,x_n]$ by **) and by convexity (since $x_n \to +\infty$ we have that $x_n > y_n$ provided $n$ is large enough). Exploiting the concave or convex-concave structure of $f_n$ in $[y_n,d_n]$ it is also easy to prove (see the Figure 1 and the proof of the previous Lemma for the details of the argument) that

$$
\tilde{f}_n(t) \leq s_n t \leq f_n(t) \quad \text{in } [y_n,d_n]. \quad (3.30)
$$

Combining (3.28), (3.29), and (3.30), we obtain that $\psi^k_\delta \leq f_n$ in $[0,d_n]$ for every $n > k$ and so, in particular,

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.pdf}
\caption{The construction of $\psi^k_\delta$ in the case $g^\infty (1) > b^0(1)$.}
\end{figure}

\[ f_n(u_n') \geq \psi^k_\delta(u_n') \] almost everywhere for $n > k$. Finally, choosing a sequence $\delta_n \downarrow 0$, by diagonalization, from the family $(\psi^k_\delta)_{k,n}$ we can extract a subfamily $(\psi^k_\delta)_{k,n}$ having all the required properties. If $g$ does not satisfy (3.26), we can proceed in the following way: let $x_0$ be the last point where $g$ vanishes and define

$$
g_k(x) := \begin{cases} 0 & \text{if } x \in [0,x_0], \\
g \left( x - \frac{1}{k} \right) & \text{if } x \geq x_0;
\end{cases}
$$
It turns out that $g_k(1) = g(1)$, $g_k \uparrow g$ as $k \to \infty$, and $g_k$ satisfies (3.26). Hence we can repeat the construction above for every $g_k$ and conclude by diagonalization. If $g$ does not satisfy (3.25), then in particular $g(t) \geq b^0(t)\mathcal{L}$ for every $t > 0$ and therefore $g_1 = b^0(1)\mathcal{L}$; moreover there exists $\pi \in \mathbb{N}$ such that $f_n(0+) > (1 - \delta)b^0(1)$ for every $n \geq \pi$. If $(d_n)$ and $(s_n)$ are as above, for $k > \pi$ we define $\psi^k_{\delta}$ by induction in the following way:

$$\psi^k_{\delta}(t) = (1 - \delta)s_k t \text{ in } [0,d_k]$$

and

$$\psi^k_{\delta}(t) = \psi^k_{\delta}(d_k + j) + \left(1 - \frac{\delta}{j + 1}\right)s_{j+1}(t - d_k + j) \text{ in } [d_k + j, d_k + j + 1] \text{ for } j \geq 0.$$

Arguing as above, it is easy to see that from the family $(\psi^k_{\delta})_{k,\delta}$ we can extract by diagonalization a subfamily satisfying all the requirements.

**Case:** $b^0(1) \geq g(1)$. Note that in this case $g_1 = g$. As above it is not restrictive to suppose that $g(0) = f_n(0) = 0$ for every $n \in \mathbb{N}$ and that $g$ satisfies (3.26). At first we choose a sequence $\delta_n \downarrow 0$ and, as above, a divergent sequence $d_n$ satisfying $\|u_n'\|_{\infty} \leq d_n$ for every $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \frac{f_n(d_n)}{d_n} = b^0(1) \geq g(1).$$

Recalling (3.8) we can define for every $i \in \mathbb{N}$

$$n_{0,i} := \inf \left\{ j \in \mathbb{N} : j > i, f_n(t) \geq (1 - \delta)g(t) \text{ in } [0, d_i], f_n'(d_i) > (1 - \delta)g'(d_i), \right.$$

$$\text{and } \frac{f_n(d_n)}{d_n} \geq (1 - \delta)g'(d_i), \forall n \geq j \left\},
$$

and, for $h \geq 1$,

$$n_{h,i} := \inf \left\{ j > n_{h,i-1} : f_n(t) \geq (1 - \delta_i + h)g(t) \text{ in } [0, d_{i+h}], f_n'(d_{i+h}) > (1 - \delta_i + h)g'(d_{i+h}), \right.$$

$$\text{and } \frac{f_n(d_n)}{d_n} \geq (1 - \delta_i + h)g'(d_{i+h}), \forall n \geq j \left\}.$$

We define the function $\psi_i$ by induction on $h$ in the following way:

$$\psi_i(t) := \begin{cases} (1 - \delta)g(t) & \text{if } t \in [0, d_i], \\ (1 - \delta_i)g(d_i) + g'(d_i)(t - d_i) & \text{if } t \in (d_i, d_{i+1}]. \end{cases}$$

and, for $h \geq 1$,

$$\psi_i(t) := \psi_h(d_{n_{h,i}}) + (1 - \delta_i + h)g'(d_{i+h})(t - d_{n_{h,i}}) \text{ in } (d_{n_{h,i}}, d_{n_{h,i+1}}];$$

clearly $\psi^\infty_i = g(1)$ for every $i$ and $\psi_i \uparrow g$ as $i \to \infty$. Set, for every $h \geq 0$,

$$\phi_{i+h}(t) := \begin{cases} (1 - \delta_i + h)g(t) & \text{if } t \in [0, d_{i+h}], \\ (1 - \delta_i + h)g(d_{i+h}) + g'(d_{i+h})(t - d_{i+h}) & \text{if } t > d_{i+h}. \end{cases}$$

First of all, taking into account the definition of $n_{h,i}$ and exploiting the structure assumption on $f_n$ exactly as we did before, one can prove that

$$\phi_{i+h} \leq f_n \quad \forall n \geq n_{i,i}; \quad (3.31)$$

moreover we have

$$\psi_i \leq \phi_{i+h} \quad \text{in } [0, d_{n_{h,i+1}}]. \quad (3.32)$$
The last inequality is an immediate consequence of the following one
\[ \psi_i \leq (1 - \delta_{i+h})g \quad \text{in } [0,d_{i+h}], \]
which can be proved easily by induction on \( h \).

Take \( n \geq n_{0,i} \) and let \( h \) be such that \( n_{h,i} \leq n \leq n_{h+1,i} \); combining (3.31) and (3.32) we finally obtain that
\[ \psi_i \leq f_n \quad \text{in } [0,d_n]. \]
\[ \square \]

**Lemma 3.11** Suppose that also (3.10) and condition iii) of Theorem 3.2 hold and let \((u_n)_{n \in \mathbb{N}} \subset W^{2,2}(I)\) be such that \( \sup_n F_n(u_n) < +\infty \). Then, for every \( \delta > 0 \), there exists a sequence \((v_n)_{n \in \mathbb{N}} \subseteq SBV(I)\) such that
\[ \|u_n - v_n\|_1 \to 0, \quad r_n \|u'_n\|_\infty \to 0 \quad \text{as } n \to \infty, \quad v'_n \leq |u'_n| \quad \text{everywhere, and} \]
\[ F_n(u_n) \geq (1 - \delta) \sum_{x \in \mathcal{X}_n} \varphi(v^+_n(x) - v_n(x)), \]
for \( n \) sufficiently large.

**Proof.** By Lemma 3.8 there exists \( K > 0 \) such that
\[ r_n \|u'_n\|_\infty \leq K; \]
for every \( 0 < s \leq K \) we define
\[ \omega_n(s) := \sup_{t \in [s,K]} \left| r_n f_n \left( \frac{t}{r_n} \right) - b(t) \right|. \]
Recalling that if a family of monotone functions pointwise converges to a continuous function, then the convergence is actually uniform on compact subsets, by (3.6) we have that \( \omega_n \to 0 \) pointwise. As a first step we choose a sequence \((c_n)_n\) of positive real numbers converging to 0 so slowly that:

a) \[ \frac{r_n}{(c_n)^2} \to 0 \quad \text{as } n \to \infty; \]

b) \[ \lim_{n \to \infty} \frac{\omega_n(c_n)}{b(c_n)} = 0. \]

We set \( D_n := \{ x \in I : |u'_n(x)| > c_n/r_n \} = \bigcup_{k=1}^{\infty} I^k_n \) \( \subseteq \), where \((I^k_n)\) is the collection of the connected components of \( D_n \); we also denote \( I^k_n = (a^k_n, b^k_n) \). Arguing as in Lemma 3.8 and taking into account condition b), we obtain
\[ |D_n| \leq \left( \frac{2 \sup_n F_n(u_n)}{b(c_n)} \right) r_n, \]
for \( n \) large enough. For every \( n \in \mathbb{N} \) we define
\[ \tilde{v}_n(x) := \begin{cases} u_n(x) & \text{if } x \in I \setminus D_n, \\ u_n(a^k_n) & \text{if } x \in (a^k_n, b^k_n); \end{cases} \]
clearly \( \tilde{v}_n \in L^1(I) \cap SBV(I) \). Moreover we set \( w_n := u_n - \tilde{v}_n \); since \( w'_n = u'_n \) and \( w''_n = u''_n \) on \( D_n \), we have
\[ F_n(u_n, I^k_n) = \int_{I^k_n} f_n(|w'_n|) \, dx + (r_n)^3 \int_{I^k_n} |w''_n|^2 \, dx; \]
summing over \( k \) and setting \( \tilde{z}_n(x) := w_n(r_n x) \) we therefore obtain
\[ F_n(u_n, D_n) = \sum_k \left( \int_{I^k_n} f_n(|w'_n|) \, dx + (r_n)^3 \int_{I^k_n} |w''_n|^2 \, dx \right) \]
\[ = \sum_k \left( \int_{I^k_n} f_n \left( r_n \frac{|z'_n(x)|}{r_n} \right) \, dx + \frac{1}{r_n} \int_{I^k_n} \frac{|z''_n(x)|}{r_n} \, dx \right) \]
\[ = \sum_k \left( r_n \int_{I^k_n} f_n \left( \frac{1}{r_n} \frac{|z'_n|}{r_n} \right) \, dy + \frac{1}{r_n} \int_{I^k_n} \frac{|z''_n|^2}{r_n} \, dy \right). \]
By (3.33) we have
\[ c_n \leq |z'_n| \leq K \quad \text{in } D_n/r_n; \quad (3.37) \]
mmoreover, we can prove that for every \( \delta > 0 \), there exists \( \pi \) such that \( r_n f_n \left( \frac{t}{r_n} \right) \geq (1 - \delta) b(t) \) for every \( t \in [c_n, K] \) and for every \( n \geq \pi \) and thus, by (3.37),
\[ r_n f_n \left( \frac{|z'|_n}{r_n} \right) \geq (1 - \delta) b(|z'|_n) \quad \text{in } D_n/r_n. \quad (3.38) \]
Indeed, by condition b), for every \( \delta > 0 \) we can find \( \pi \) such that \( \omega_n(c_n) \leq \delta b(c_n) \) for every \( n \geq \pi \), so that, recalling (3.34),
\[ r_n f_n \left( \frac{t}{r_n} \right) \geq b(t) - \omega_n(c_n) \geq b(t) - \delta b(c_n) \geq (1 - \delta) b(t) \quad \forall t \in [c_n, K], \]
where we used the monotonicity of \( b \). Let us define the functions \( z_n \) as
\[ z_n(x) := \begin{cases} \frac{\bar{z}_n(x)}{r_n} & \text{if } x \in (I \setminus D_n)/r_n, \\ \frac{\bar{z}_n(x) - z_n \left( \frac{a_n^k}{r_n} \right) (x - a_n^k)}{r_n} & \text{in } I_n/r_n. \end{cases} \]
By (3.36) and (3.38), by using the fact that \( |z'_n| \leq |z'|_n \) everywhere and the monotonicity of \( b \), we have
\[ F_n(u_n, D_n) \geq (1 - \delta) \sum_k \left( \int_{I_n/r_n} b(|z'_n|) \, dy + \int_{I_n/r_n} |z''_n|^2 \, dy \right) \geq (1 - \delta) \sum_k \left( \int_{I_n/r_n} b(|z'_n|) \, dy + \int_{I_n/r_n} |\bar{z}'_n|^2 \, dy \right) \geq (1 - \delta) \sum_k \varphi \left( \left| z_n \left( \frac{b_n^k}{r_n} \right) \right| \right) \left( 1 - \delta \right) \sum_k \varphi \left( v_n^\pm(b_n^k) - v_n(b_n^k) \right) = (**), \quad (3.39) \]
where \( v_n(x) := u_n(x) - z_n(x/r_n) \). Using the definition of \( z_n \), it is easy to check that
\[ v_n(x) = \begin{cases} u_n(x) & \text{if } x \in I \setminus D_n \\ u_n(a_n^k) + u'_n(a_n^k)(x - a_n^k) & \text{if } x \in (a_n^k, b_n^k) \end{cases} \]
and to see that \( (** \rightleftharpoons (1 - \delta) \sum_{x \in S_n} \varphi (v_n^\pm(x) - v_n(x)), \) which, combined with (3.39), gives the thesis of the lemma, once we have shown that
\[ \|v_n - u_n\|_1 \to 0 \quad \text{as } n \to \infty. \quad (3.40) \]
If \( t \in I_n^k \), by Hölder’s Inequality, we have
\[ |v_n(t) - u_n(t)| \leq \int_{a_n^k}^t |u'_n(s) - u'_n(s)| \, ds \leq \int_{a_n^k}^t \int_{a_n^k}^t |u''_n(z)| \, dz \leq \left( \int_{I_n^k} |u''_n|^2 \, dz \right)^{\frac{1}{2}} \left( \int_{a_n^k}^t (s - a_n^k) \frac{1}{2} \, ds \right) = \frac{2}{3} \left( \int_{I_n^k} |u''_n|^2 \, dz \right)^{\frac{1}{2}} \left( t - a_n^k \right)^{\frac{3}{2}}, \]
therefore, integrating on $I_n$,

$$\int_{I_n} |v_n(t) - u_n(t)| \, dt \leq \frac{4}{15} \left( \int_{I_n} |u_n'|^2 \, dz \right)^{\frac{1}{2}} |I_n|^{\frac{1}{2}};$$

using (3.35), we can conclude

$$\|u_n - v_n\|_1 \leq \frac{4}{15} \sum_{k \in \mathbb{N}} \left( \int_{I_n} |u_n'|^2 \, dz \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{N}} |I_n|^k \right)^{\frac{1}{2}} \leq \frac{4}{15(r_n)^{\frac{1}{2}}} \left( \int_{D_n} |u_n'|^2 \, dz \right)^{\frac{1}{2}} \leq \frac{4}{15(r_n)^{\frac{1}{2}}} \frac{(\sqrt{2})^3}{\sqrt{2}} \left( \int_{D_n} |u_n'|^2 \, dz \right)^{\frac{1}{2}} \leq \frac{4}{15} \left( \sup_n F_n(u_n) \right)^{\frac{1}{2}} \frac{r_n}{(b(c_n))^{\frac{1}{2}}};$$

recalling condition a) and (3.10), we finally get (3.40).

\[\square\]

**Lemma 3.12** Let $g : [0, \infty) \to [0, \infty)$ be a convex superlinear function and let $u \in SBV(I)$ be such that $\int_I g(|u'|) \, dx + H^0(S_u) < +\infty$. Then there exists a sequence $(u_n) \in SBV(I)$ such that $S_{u_n} \subseteq S_u$, $u_n \in W^{2,2}(I \setminus S_u)$, $u_n(t) = 0$ on $S_u$, $u_n \to u$ in $L^\infty(I)$, $u_n(t) \to u^\pm(t)$ on $S_u$, and $\int_I g(|u_n'|) \, dx \to \int_I g(|u'|) \, dx$.

**Proof.** Let $I = (a, b)$ and $S_u = \{x_1, \ldots, x_N\}$ with $x_i < x_{i+1}$ and set $x_0 = a$ and $x_{N+1} = b$. We can construct a family $(g_k)$ of strictly convex and superlinear functions belonging to $C^2([0, +\infty))$ and satisfying

$$g'_k(0) = 0, \quad g_k \downarrow g, \quad \text{and} \quad \lim_{t \to +\infty} \frac{g_k(t)}{g(t)} = 1.$$ 

For every $k \in \mathbb{N}$ and for every $i \in \{0, \ldots, N\}$ let $u^k_{i,j}$ be the solution of the minimum problem

$$\min \left\{ \int_I g_k(|u'|) \, dx + j \int_{x_i}^{x_{i+1}} |v - u| \, dx : v \in W^{1,1}(x_i, x_{i+1}) \right\};$$

note that the existence of such a solution is guaranteed by the convexity and the superlinearity of $g_k$; moreover $u^k_{i,j}$ is a classical solution to the Euler equation $h_k''(w)w'' = j(w - u)$ with the Neumann conditions $w'(x_i) = w'(x_{i+1}) = 0$, where $h_k$ is the function in $C^2([0, +\infty))$ obtained by reflection of $g_k$. Therefore, taking into account the regularity and strict convexity assumptions on $g_k$ and the fact that $u \in C([x_i, x_{i+1}])$, we get $u^k_{i,j} \in C^2([x_i, x_{i+1}])$, so that, denoting by $u_{i,j}^k$ the function in $SBV(I)$ which coincides with $u^k_{i,j}$ on $(x_i, x_{i+1})$, we clearly have that the family $(u^k_{i,j})$ satisfies all the required conditions except for the last one. Indeed, by construction, we know only $\int_I g_k(|u^k_{i,j}'|) \, dx \overset{\frac{1}{2}}{=} \int_I g_k(|u'|) \, dx$, but recalling $\int_I g_k(|u'|) \, dx \overset{k}{\to} \int_I g(|u'|) \, dx$, the desired approximating sequence can be obtained by diagonalization.

We finally state a Lemma which will be useful in the sequel.

**Lemma 3.13** Denote by $A(\Omega)$ the family of all open subsets of $\Omega$ and let $\nu : A(\Omega) \to [0, +\infty)$ be a superadditive set-function. Let $\lambda$ be a positive measure on $\Omega$ and let $(\psi_i)_i$ a family of positive Borel functions such that $\nu(A) \geq \int_A \psi_i \, d\lambda$ for all $A \in A(\Omega)$ and for all $i \in \mathbb{N}$. Then $\nu(A) \geq \int_A \sup \psi_i \, d\lambda$, for all $A \in A(\Omega)$.

**Proof.** See Proposition 1.16 of [10].

**Proof of Theorem 3.2:** the case $\theta^0(1) = +\infty$.

- $\Gamma$-lim sup Inequality
Let us set for notational convenience \( F'' := \Gamma\)-\( \limsup_n F_n \). We first remark that it is enough to show that \( F''(u) \leq \int_I g(|u'|) \, dx + \sum_{S_u} \varphi(u^+ - u^-) \) for every \( u \in SBV(I) \) with \( H^0(S_u) < +\infty \), indeed, once we have this inequality, the thesis follows from the semicontinuity of \( F'' \) and the fact that \( \overline{F}_{\varphi,g} \) coincides with the relaxed functional of

\[
H(u) := \begin{cases} 
\int_I g(|u'|) \, dx + \sum_{S_u} \varphi(u^+ - u^-) & \text{if } u \in SBV(I) \text{ and } H^0(S_u) < +\infty, \\
+\infty & \text{otherwise.}
\end{cases}
\]

**Claim 1.** Let \( u \in SBV(I) \) such that \( H^0(S_u) < +\infty \), \( u \in W^{2,2}(I \setminus S_u) \), \( F(u) < +\infty \), and \( u'(t) = 0 \) for every \( t \in S_u \). Then

\[
F''(u) \leq \int_I g(|u'|) \, dx + \sum_{S_u} \varphi(u^+ - u^-).
\]

Since the construction is local, we may assume that \( S_u = \{ \bar{t} \} \) and \( u(t) = u^+(\bar{t}) \).

Fix \( \delta > 0 \) and choose an admissible pair \((\eta, v)\) for problem (3.3) (with \( z = u^+(\bar{t}) - u(\bar{t}) \)) satisfying:

\[
\int_0^\eta b(|v'|) \, dx + \int_0^\eta v''^2 \, dx < \varphi(u^+(\bar{t}) - u(\bar{t})) + \delta.
\]

We define the recovery sequence in the following way:

\[
u_n(x) := \begin{cases}
    u(x) & \text{if } x \leq \bar{t} \\
    v\left(\frac{x - \bar{t}}{r_n}\right) + u(\bar{t}) & \text{if } x \in (\bar{t}, \bar{t} + r_n \eta) \\
    u(x - r_n \eta) + u^+(\bar{t}) & \text{if } x \geq \bar{t} + r_n \eta.
\end{cases}
\]

Clearly \( u_n \to u \) in \( L^1 \). We can now compute

\[
F_n(u_n) = F_n(u_n, I \setminus (\bar{t}, \bar{t} + r_n \eta)) + \int_{\bar{t}}^{\bar{t} + r_n \eta} f_n\left(\frac{1}{r_n}, v\left(\frac{x - \bar{t}}{r_n}\right)\right) \, dx \\
+ (r_n)^3 \int_{\bar{t}}^{\bar{t} + r_n \eta} \frac{1}{(r_n)^2} \left| v''\left(\frac{x - \bar{t}}{r_n}\right)\right|^2 \, dx \\
= F_n(u_n, I \setminus (\bar{t}, \bar{t} + r_n \eta)) + \int_0^\eta r_n f_n\left(\frac{v'}{r_n}\right) \, dy + \int_0^\eta \left| v'' \right|^2 \, dy.
\]

Since

\[
r_n f_n\left(\frac{v'}{r_n}\right) \to b(v') \text{ in } \{x \in I : \left| v'(x) \right| \neq 0\},
\]

by the Dominated Convergence Theorem and the fact that

\[
\lim_{n \to \infty} \int_{\{x \in I : |v'(x)| \neq 0\}} r_n f_n\left(\frac{v'}{r_n}\right) \, dx = \lim_{n \to \infty} \int_{\{x \in I : |v'(x)| \neq 0\}} |v'(x)| \, dx = 0,
\]

we have

\[
\limsup_{n \to \infty} \left(\star_n\right) \leq \int_0^\eta b(|v'|) \, dx + \int_0^\eta |v''|^2 \, dx \quad \text{as } n \to \infty;
\]

moreover, again by the Dominated Convergence Theorem, we easily see that

\[
\lim_{n \to \infty} F_n(u_n, I \setminus (\bar{t}, \bar{t} + r_n \eta)) = \int_I g(|u'|) \, dx.
\]
From (3.41), we therefore obtain
\[
\limsup_{n \to \infty} F_n(u_n) \leq \int_I g(|u'|) \, dx + \int_0^\eta b(|u'|) \, dx + \int_0^\eta |u''|^2 \, dx
\leq \int_I g(|u'|) \, dx + \varphi(u^+(\bar{t}) - u(\bar{t})) + \delta.
\]

By the arbitrariness of \( \delta \), Claim 1 is proved. By a standard density argument based on the use of Lemma 3.12 we recover the same inequality for every \( u \in SBV(I) \) with \( \mathcal{H}^0(S_u) < +\infty \) and this concludes the proof of the \( \Gamma \)-limsup inequality, as we remarked above.

- \( \Gamma \)-liminf Inequality

We are assuming that the structure condition iii) holds so that \( g \) is convex and \( b \) is convex or concave or convex-concave; the functional \( F_{b,\varphi} \) is then well defined and, by Theorem 2.3, coincides with \( F_{b,\varphi}^\Gamma \). We distinguish two cases.

Case 1: \( g^\infty(1) = +\infty \) i.e. \( g \) is superlinear.

Note that in this case \( F_{b,\varphi}(u) \) is finite only if \( u \in SBV(I) \) and for such \( u \) we have
\[
F_{b,\varphi}(u) = \int_I g(|u'|) \, dx + \sum_{x \in S_u} \varphi(u^+ - u^-).
\]

Let \( u_n \to u \) in \( L^1 \) and such that \( \sup_n F_n(u_n) < +\infty \), let \( (v_n)_n \) be the sequence constructed in Lemma 3.11 and \( (\psi_i)_i \) the associated family of convex superlinear functions according to Lemma 3.10. For \( \delta \) and \( \mu \in (0,1) \), and for every open subset \( J \subseteq I \), by Lemmas 3.11 and 3.10, we have, for \( n \) sufficiently large,
\[
F_n(u_n, J) = (1 - \delta) \int_J f_n(u_n') \, dx + \delta \left[ \int_J f_n(|u_n'|) \, dx + ((1/\sqrt{\delta}) r_n)^3 \int_J u_n''^2 \, dx \right]
\geq (1 - \delta) \int_J \psi_i(|u_n'|) \, dx + \delta (1 - \mu) \sum_{x \in S_u} \varphi(v_n^+ - v_n^-). \tag{3.42}
\]

Therefore, by the Ambrosio Semicontinuity Theorem (recall also Lemma 3.6), we obtain that \( u \in SBV(I) \) and
\[
\liminf_{n \to \infty} F_n(u_n, J) \geq (1 - \delta) \int_J \psi_i(u'_i) \, dx + \delta (1 - \mu) \sum_{x \in S_u} \varphi(u^+ - u^-) \quad \forall i;
\]

letting \( i \uparrow \infty \) and \( \mu \downarrow 0 \), we obtain
\[
(\Gamma \text{-} \liminf_{n \to \infty} F_n)(u, J) \geq (1 - \delta) \int_J g(|u'|) \, dx + \delta \sum_{x \in S_u} \varphi(u^+ - u^-)
\]
\[
= \int J h^\delta(x) \, d\lambda \quad \forall \text{ open } J \subseteq I, \forall \delta \in (0,1), \tag{3.43}
\]
where we have set \( \lambda := g(|u'|) \mathcal{L}^1 + \varphi(u^+ - u^-) \mathcal{H}^0 \) and \( h^\delta := (1 - \delta)(1 - \chi_{S_u}) + \delta \chi_{S_u} \). Let \( \delta_n \) be a dense sequence in \((0,1)\); since \( \sup_i h^\delta_i = 1 \), from (3.43) we finally deduce
\[
(\Gamma \text{-} \liminf_{n \to \infty} F_n)(u) \geq \int_I \sup_i h^\delta_i \, d\lambda = \int_I g(|u'|) \, dx + \sum_{x \in S_u} \varphi(u^+ - u^-),
\]
where we applied Lemma 3.13 (with \( \nu := (\Gamma \text{-} \liminf_{n \to \infty} F_n)(u, \cdot) \)).

Case 2: \( g^\infty(1) < +\infty \). Let \( v_n \) be as above: according to Lemma 3.10, construct a family \( (\psi_i)_i \) of convex functions such that \( \psi_i^\infty(1) = g^\infty(1) \) for every \( i \in \mathbb{N} \), \( \psi_i \uparrow g \) as \( i \to \infty \), and \( \psi_i(|v_n'|) \leq f_n(|v_n'|) \) for every \( i \) and for \( n \) sufficiently large. Therefore, by using Lemma 3.11, we can write
\[
F_n(u_n, I) = F_n(u_n, D_n) + F_n(u_n, I \setminus D_n)
\geq (1 - \delta) \left[ \int_I \psi_i(|v_n'|) \, dx + \sum_{x \in S_u} \varphi(v_n^+ - v_n^-) \right] - \int_{D_n} \psi_i(|v_n'|) \, dx; \tag{3.44}
\]
using the inequality $\psi_i(t) \leq g(0) + g^\infty(1)t$, true for every $t > 0$, and recalling (3.35) and the fact that $\lim_n b(c_n)/c_n = +\infty$, we can estimate
\[\int_{D_n} \psi_i(\|u_i\|) \, dx = \psi_i \left(\frac{c_n}{r_n}\right) |D_n| \leq \left(g(0) + g^\infty(1)\frac{c_n}{r_n}\right) \left(\frac{2 \sup_n F_n(u_n)}{b(c_n)}\right) r_n = O(1)\]
hence, invoking the Relaxation Theorem 2.3, from (3.44), we get
\[\liminf_{n \to \infty} F_n(u_n) \geq \left(1 - \delta\right) \left(\int_I \psi_i(|u'|) \, dx + \sum_{s_n} (\varphi \triangle g^\infty)(u^+ - u^-) + g^\infty(1) |D^c u|\right).\]
Letting $i \uparrow +\infty$ and $\delta \downarrow 0$ we complete the proof of the $\Gamma$-liminf inequality.

Concerning the last part of the theorem, we first observe that, thanks to (3.17) the approximating functionals are equicoercive: the conclusion then follows from Remark 3.9. \hfill \Box

To treat the case
\[b^0(1) = \lim_{t \to +0} \frac{b(t)}{t} < +\infty,\]
we need first the following lemma.

**Lemma 3.14** Let $b$ be satisfy (3.14), and, for every $\delta > 0$, let $\varphi^\delta : (0, \infty) \to (0, \infty)$ be the function defined by
\[\varphi^\delta(z) := \inf_{\eta > 0} \left\{ \int_0^\eta b(|u'|) \, dx + \int_0^\eta |u''|^2 \, dx : u \in W^{2,2}(0, \eta), \quad u(0) = 0, u(\eta) = z, u'(0) = u'(\eta) = \delta \right\}.\]

Then the following properties hold true:

i) $\lim_{\delta \to 0^+} \varphi^\delta(z) = \varphi(z)$, uniformly in $|k, +\infty)$, for every $k > 0$;

ii) for every $\varepsilon \in (0, 1)$, there exists $\delta$ such that $\varphi^\delta(z) \geq (1 - \varepsilon)\varphi(z)$ for every $\delta \leq \delta$ and for every $z > 0$.

**Proof.** Fix $k > 0$ and let $\phi \in C^2([0, 1])$ be such that $\phi(0) = \phi(0) = 0$ and $\phi(1) = \phi'(1) = 1$. Moreover choose $0 < \delta < k$ such that
\[\int_0^1 b(\delta |\phi'|) \, dx + \delta^2 \int_0^1 |\phi''|^2 \, dx \leq \frac{\varepsilon}{4},\]
for every $\delta \leq \delta$. Fix now $\delta \in (0, \delta)$ and for a given $z \geq k$ set $z' := z - 2\delta$ and take $(v, \eta)$, admissible pair for the minimum problem defining $\varphi(z')$ such that
\[\int_0^\eta b(|v'|) \, dx + \int_0^\eta |v''|^2 \, dx \leq \varphi(z') + \frac{\varepsilon}{2} \leq \varphi(z) + \frac{\varepsilon}{2}.\]

We now define $\tilde{\eta} := \eta + 2$ and $\bar{\delta} \in W^{2,2}(0, \tilde{\eta})$ by
\[\bar{\delta}(t) := \begin{cases} \delta - \delta \phi(1 - t) & \text{if } t \in [0, 1), \\ \psi(t - 1) + \delta & \text{if } t \in [1, \eta + 1), \\ z - \delta + \delta \phi(t - \eta - 1) & \text{if } t \in [\eta + 1, \tilde{\eta}]. \end{cases}\]

It is clear that $(\bar{\delta}, \tilde{\eta})$ is an admissible pair for the minimum problem defining $\varphi^\delta(z)$, so that we have
\[\varphi^\delta(z) \leq \int_0^\eta b(|\bar{\delta}'|) \, dx + \int_0^\eta |\bar{\delta}''|^2 \, dx \]
\[= 2 \left(\int_0^1 b(\delta |\phi'|) \, dx + \delta^2 \int_0^1 |\phi''|^2 \, dx\right) + \int_0^\eta b(|v'|) \, dx + \int_0^\eta |v''|^2 \, dx \]
\[\leq \varphi(z) + \varepsilon, \quad (4.47)\]
where the last inequality follows from (3.45) and (3.46). Let now \((v, \eta)\) be an admissible pair for \(\varphi^\delta (z)\) satisfying
\[
\int_0^\eta b(|v'|) \, dx + \int_0^\eta |v''|^2 \, dx \leq \varphi^\delta (z) + \frac{\varepsilon}{2}.
\]  
(3.48)
We now define \(\tilde{\eta} := \eta + 2\) and \(\delta\) by
\[
\tilde{\varphi}(t) := \begin{cases} 
\delta \varphi(t) & \text{if } t \in [0,1), \\
v(t-1) + \delta & \text{if } t \in [1, \eta + 1), \\
z + 2\delta - \delta \varphi(\tilde{\eta} - t) & \text{if } t \in [\eta + 1, \tilde{\eta}].
\end{cases}
\]
As above, we have
\[
\varphi(z) \leq \varphi(z + 2\delta) \leq \int_0^\delta b(|\varphi'|) \, dx + \int_0^\delta |\varphi''|^2 \, dx
\]
\[
= 2 \left( \int_0^1 b(\delta |\varphi'|) \, dx + \delta^2 \int_0^1 |\varphi''|^2 \, dx \right) + \int_0^\delta b(|v'|) \, dx + \int_0^\delta |v''|^2 \, dx
\]
\[
\leq \varphi^\delta (z) + \varepsilon,
\]
thanks to (3.45) and (3.48); recalling (3.47), \(i\) is proved.
For the last part we suppose by contradiction that there exist \(\varepsilon \in (0,1)\), a sequence \(\delta_n \downarrow 0\), and sequence \(x_n\) such that
\[
\varphi^{\delta_n}(x_n) < (1 - \varepsilon)\varphi(x_n),
\]
(3.49)
for every \(n \in \mathbb{N}\). Testing with the pair \((v(t) := \delta t, z/\delta)\), we easily obtain
\[
\varphi^\delta (z) \leq \frac{b(\delta)}{\delta} z \leq C' z, \quad \forall \delta < 1.
\]
(3.50)
Taking into account \(i\) we see that (3.49) and (3.50) imply
\[
x_n \to 0 \quad \text{and} \quad \varphi^{\delta_n}(x_n) \to 0.
\]
(3.51)
Let \((v_n, \eta_n)\) be an admissible pair for the minimum problem defining \(\varphi^{\delta_n}(x_n)\) such that
\[
\int_0^{\eta_n} b(|v'_n|) \, dx + \int_0^{\eta_n} |v''_n|^2 \, dx \leq \varphi^{\delta_n}(x_n) + (\varphi^{\delta_n}(x_n))^2;
\]
arguing as in the proof of Lemma 3.6 we deduce that \(\|v'_n\|_{\infty} \to 0\). Choose \(\sigma > 0\) such that
\[
b(t) \geq \left( 1 - \frac{\varepsilon}{2} \right) C t \quad \forall t \leq \sigma
\]
(3.53)
and let \(\pi\) be such that \(\|v'_n\|_{\infty} \leq \sigma\) for every \(n \geq \pi\). Then, using (3.52), (3.53), (3.50), (3.51) and recalling that \(\varphi(z) \leq C z\) for every \(z > 0\) (see (3.15)), we estimate
\[
\varphi^{\delta_n}(x_n) \geq \int_0^{\eta_n} b(v'_n) \, dx - (\varphi^{\delta_n}(x_n))^2 \geq \left( 1 - \frac{\varepsilon}{2} \right) C x_n - (C' x_n)^2
\]
\[
\geq \left( 1 - \frac{3}{4} \varepsilon \right) C x_n \geq \left( 1 - \frac{3}{4} \varepsilon \right) \varphi(x_n),
\]
for \(n\) large enough, a contradiction with (3.49).

We are now in a position to conclude the proof of Theorem 3.2.

PROOF OF THEOREM 3.2: the case \(b^\theta (1) < +\infty\).
The $\Gamma$-limsup inequality can be proved as in the other case. So suppose that the structure condition iii) appearing in the statement of the theorem holds so that $F_{\delta, g}$ is well defined and coincides with $F_{\delta, g}$. We may also suppose that $g \neq 0$, otherwise the $\Gamma$-liminf inequality is trivial. Let $\varepsilon_n \to 0$ and $u_{\varepsilon_n} \to u$ in $L^1$ and such that $\exists \lim_{n \to \infty} F_{\varepsilon_n}(u_{\varepsilon_n}) < +\infty$. Choose now an infinitesimal sequence $c_n$ with the same properties as in the proof of Lemma 3.11; set

$$
D_n := \left\{ x \in I : |u'_{\varepsilon_n}| > \frac{c_n}{r(\varepsilon_n)} \right\} = \bigcup_{k=1}^{\infty} \left\{ b_n^k, b_n^k \right\} = \bigcup_{k=1}^{\infty} V_k^n
$$

and define

$$
v_{\varepsilon_n}(x) := \left\{ \begin{array}{ll}
    u_{\varepsilon_n}(x) & \text{if } x \in I \setminus D_n \\
    \alpha_n(x) & \text{if } x \in (b_n^k, b_n^k) 
\end{array} \right.
$$

Finally set $w_{\varepsilon_n} := u_{\varepsilon_n} - v_{\varepsilon_n}$ and $z_{\varepsilon_n}(x) := (\varepsilon_n(r(x)n)).$ For fixed $\delta \in (0,1)$, with exactly the same arguments of Lemma 3.11, we obtain

$$
F_{\varepsilon_n}(u_{\varepsilon_n}, D_n) \geq (1 - \delta) \sum_k \left( \int_{I_k^\varepsilon_n} b(\varepsilon_n|z'|^2) dx + \int_{I_k^\varepsilon_n} |z''|^2 dx \right)
$$

$$
\geq (1 - \delta) \sum_k \inf \inf \left\{ \int_0^\eta b(|z'|) dx + \int_0^\eta |z''|^2 dx : z \in W^{2,2}(0, \eta), 
\right. 
\left. z(0) = 0, z(\eta) = |w_{\varepsilon_n}(b_n^k)|, \right.
\left. z'(0) = z'(\eta) = c_n \right\}
$$

$$
\geq (1 - \delta) \sum_k \varphi^\omega_0 (|w_{\varepsilon_n}(b_n^k)|) = (1 - \delta) \sum_{s_{\varepsilon_n}} \varphi^\omega (v_{\varepsilon_n} - v_{\varepsilon_n}), \quad (3.54)
$$

for $n$ large enough, where $\varphi^\omega$ is the function defined in Lemma 3.14 (with $\delta = c_n$). Using ii) of Lemma 3.14, from (3.54) we deduce

$$
F_{\varepsilon_n}(u_{\varepsilon_n}, D_n) \geq (1 - \delta)^2 \sum_{s_{\varepsilon_n}} \varphi(v_{\varepsilon_n} - v_{\varepsilon_n}),
$$

for $n$ large enough. Combining the estimate above with Lemma 3.10, we therefore obtain (passing to a subsequence, if needed),

$$
F_{\varepsilon_n}(u_{\varepsilon_n}) \geq (1 - \delta)^2 \left( \int_{I \setminus D_n} \varphi_1(|u'|^2) dx + \sum_{s_{\varepsilon_n}} \varphi(v_{\varepsilon_n} - v_{\varepsilon_n}) \right)
$$

$$
= (1 - \delta)^2 \left( \int_I \varphi_1(|u'|) dx + \sum_{s_{\varepsilon_n}} \varphi(v_{\varepsilon_n} - v_{\varepsilon_n}) \right), \quad (3.55)
$$

where, according to Lemma 3.10, $\varphi_1$ is convex, $\psi^\infty_1(1) = g^\infty_1(1) \wedge b^0(1)$ and $\psi_1 \uparrow g_1$ as $i \to \infty$. Since, by Lemma 3.8, we have $\sup_n \operatorname{Var} v_{\varepsilon_n} \leq \sup_n \operatorname{Var} u_{\varepsilon_n} < +\infty, \operatorname{Rellich}$'s theorem implies that $v_{\varepsilon_n}$ is precompact in $L^1$ and since $v_{\varepsilon_n} \to u$ in measure (recall that $|D_n| \to 0$), we get $v_{\varepsilon_n} \to u$ in $L^1$. Applying Theorem 2.3 (recall that $b^0(1) = \varphi^0(1)$, by virtue of Lemma 3.6), from (3.55) we deduce

$$
\liminf_{n \to \infty} F_{\varepsilon_n}(u_{\varepsilon_n}) \geq (1 - \delta)^2 \left( \int_I \varphi_1(|u'|) dx + \sum_{s_{\varepsilon_n}} \varphi_1(u^+ - u) \right) + \left( g^\infty (1) \wedge b^0(1) \right) |D^c u|;
$$

letting $i \uparrow +\infty$ and $\delta \downarrow 0$ we finally obtain the desired $\Gamma$-liminf inequality. \qed
Remark 3.15 Looking carefully at the proof we see that the structure assumption iii) of Theorem 3.2 can be slightly weakened without changing the result; more precisely it is sufficient to suppose that there exists a family \((g^k_n)_{n,k}\) of positive continuous non-decreasing functions enjoying the following properties:

i) \(f_n \geq g^k_n\) for every \(n, k \in \mathbb{N}\);

ii) for every \(k \in \mathbb{N}\) the family \((g^k_n)\) satisfies either st1) or st2);

iii) \(g^k_n(t) \to g^k(t)\) for every \(t \geq 0\) and \(r_n g^k_n(t/r_n) \to b^k(t)\) for every \(t > 0\), as \(n \to \infty\), with \(g^k\) and \(b^k\) satisfying

\[ g^k \uparrow g \quad \text{and} \quad (b^k)'(1) \uparrow b^0(1) \quad \text{as } k \to \infty. \]

Indeed call \(G^k_n\) the functional associated with \(g^k_n\); then, for every \(k \in \mathbb{N}\), by Theorem 3.2 we have

\[ \Gamma - \lim_{n \to \infty} F_n \geq \Gamma - \lim_{n \to \infty} G^k_n = F_{b^*,g^*}, \]

where \(F_{b^*,g^*} \uparrow F_{b,g}\) as \(k \to \infty\).

We want now to show that if \(g : [0, +\infty) \to [0, +\infty)\) is any superlinear non-decreasing convex function and \(b : [0, +\infty) \to [0, +\infty)\) is an arbitrary concave function with \(b^0(1) = +\infty\), then \(F_{b,g}\) can be reached by functionals of the form (3.1).

Theorem 3.16 Let \(g : [0, +\infty) \to [0, +\infty)\) be non-decreasing, convex, and superlinear \((g^\infty(1) = +\infty)\) and let \(b : [0, +\infty) \to [0, +\infty)\) be non-decreasing and concave with \(b(0) = 0\) and \(b^0(1) = +\infty\). Then there exists a family \((f_\varepsilon)\) of positive, continuous, and non-decreasing functions such that the functionals

\[ F_\varepsilon := \begin{cases} \int_I f_\varepsilon(u') \, dx + \varepsilon^{\frac{3}{2}} \int_I |u''|^2 \, dx & \text{if } u \in W^{2,2}(I); \\ +\infty & \text{otherwise in } L^1(I), \end{cases} \]

\(\Gamma\)-converge with respect to the \(L^1\)-metric to \(F_{b,g}\), as \(\varepsilon \to 0^+\).

The theorem is an immediate consequence of Theorem 3.2 and of the following proposition which is proved in [22] (see Lemmas 6.6 and 6.7).

Proposition 3.17 Let \(g\) and \(b\) be as in the previous theorem. Then the functions \(f_\varepsilon\) defined by

\[ f_\varepsilon(t) := \min \left\{ g(s) + \frac{1}{\varepsilon} b(\varepsilon(t - s)) : s \in [0,t] \right\}, \]

are continuous, non-decreasing and satisfy the following properties:

i) \(f_\varepsilon(t) \to g(t)\) for every \(t \geq 0\);

ii) \(\varepsilon f(\varepsilon t) \to b(t)\) for every \(t > 0\);

iii) setting \(x_\varepsilon := \sup \{ x \geq 0 : f_\varepsilon(x) = g(x) \}\), there holds that \(f_\varepsilon = g\) in \([0,x_\varepsilon]\) and \(f_\varepsilon\) is concave in \([x_\varepsilon, +\infty)\); moreover \(x_\varepsilon \to +\infty\) as \(\varepsilon \to 0^+\).

We conclude this subsection with some considerations on the asymptotic behaviour of the function \(\varphi\) defined in (3.3).

Proposition 3.18 i) Let \(b(t) = ct^p\) with \(c > 0\) and \(p \in [0,1)\). Then \(\varphi(z) = m(p)c^{\frac{3}{2}} z^{\frac{3}{2}p} \), where

\[ m(p) := \min \left\{ \left( \frac{3}{1 - p} \right)^{\frac{1}{2}p} + \left( \frac{1 - p}{3} \right)^{\frac{1}{2}p} \left( \int_0^1 |v''|^2 \, dt \right)^{\frac{1}{2}p} \left( \int_0^1 |v'|^p \, dt \right)^{\frac{2}{1 - p}} : v \in W^{2,2}(0,1), v(0) = 0, v(1) = 1, v'(0) = v'(1) = 0 \right\}. \]
ii) Let \( b : [0, +\infty) \to [0, +\infty) \) be concave with \( b'(1) \neq 0 \). Then the function \( \varphi \) defined in (3.3) satisfies the growth condition
\[
C_1(\sqrt{z} - 1) \leq \varphi(z) \leq C_2(z + 1) \quad \forall z \geq 0,
\]
for suitable \( C_1, C_2 > 0 \).

iii) For every \( \gamma \in [1/2, 1) \) there exists a concave function \( b \) satisfying the hypotheses of Theorem 3.16 such that the associated \( \varphi \) satisfies
\[
\lim_{z \to +\infty} \frac{\varphi(z)}{z^\gamma} = +\infty \quad \text{and} \quad \lim_{z \to +\infty} \frac{\varphi(z)}{z^{\gamma+\varepsilon}} = 0 \quad \forall \varepsilon > 0.
\]

PROOF. i): For notational convenience we set
\[
S_{\eta,z} := \{ u \in W^{2,2}(0, \eta) : u(0) = 0, u(\eta) = z, u'(0) = u'(\eta) = 0 \};
\]
then, noting that for every \( v \in S_{\eta,z} \) we can write \( v(\cdot) = w(\cdot/\eta) \) with \( w \in S_{1,z} \), we can use the definition of \( \varphi \) to compute
\[
\varphi(z) = \inf_{\eta} \inf_{v \in S_{\eta,z}} \left( c \int_0^\eta |v'|^p \, dt + \int_0^\eta |v''|^2 \, dt \right)
= \inf_{w \in S_{1,z}} \inf_{\eta} \left( c \int_0^\eta \frac{1}{\eta^p} \left| w' \left( \frac{t}{\eta} \right) \right|^p \, dt + \int_0^\eta \frac{1}{\eta^p} \left| w'' \left( \frac{t}{\eta} \right) \right|^2 \, dt \right)
= \inf_{w \in S_{1,z}} \left\{ \left[ \left( \frac{3}{1-p} \right)^{\frac{1-p}{2p}} + \left( \frac{1-p}{3} \right)^{\frac{3}{2p}} \right] c^{\frac{3}{2p}} \left( \int_0^1 |w''|^2 \, ds \right)^{\frac{1-p}{2p}} \left( \int_0^1 |w'|^p \, ds \right)^{\frac{3}{2p}} \right\}
= \inf_{w \in S_{1,z}} \left\{ \left[ \left( \frac{3}{1-p} \right)^{\frac{1-p}{2p}} + \left( \frac{1-p}{3} \right)^{\frac{3}{2p}} \right] \left( \int_0^1 |w''|^2 \, ds \right)^{\frac{1-p}{2p}} \left( \int_0^1 |w'|^p \, ds \right)^{\frac{3}{2p}} \right\}
= m(p)c^{\frac{3}{2p}} z^{-\frac{p}{2p}}.
\]

It is clear also from the computations above that there exists an optimal pair \((\eta, v)\) for the problem defining \( \varphi \): let \( v \) be a solution of problem (3.56), then, setting
\[
\eta := \left[ \frac{3}{c(p(1-p))} \right]^{\frac{1-p}{2p}} \left( \frac{\int_0^1 |w''|^2 \, ds}{\int_0^1 |w'|^p \, ds} \right)^{\frac{3}{2p}} z^{-\frac{p}{2p}} \quad \text{and} \quad w(t) := zv \left( \frac{t}{\eta} \right),
\]
we have that \((\eta, w)\) is an optimal pair.

ii): Under our assumptions there exists \( C > 0 \) such that \( b(t) \leq C(1 + t) \) for every \( t \geq 0 \). Take \((\eta, v)\) such that \( v \in S_{\eta,z} \), \( v \) is non-decreasing and
\[
C\eta + \int_0^\eta v'' \, dx = m(0)C^{3/4}/\sqrt{z}:
\]
this is possible thanks to the previous point. Then
\[
\varphi(z) \leq C \int_0^\eta v' \, dx + C\eta + \int_0^\eta |v''|^2 \, dx = Cz + m(0)C^{3/4}/\sqrt{z} \leq C'(1 + z).
\]
Concerning the other inequality, since, under our hypotheses, there exist \( \alpha, \beta > 0 \) such that \( b(t) \geq at \land \beta \), it will be enough to prove the following claim.
Claim. Let \( b(t) = \alpha t \wedge \beta \) with \( \alpha, \beta > 0 \). Then
\[
\lim_{z \to +\infty} \frac{\varphi(z)}{m(0)\beta^{3/4}\sqrt{z}} = 1.
\]

First of all, since \( b(t) \leq \beta \), by comparison and by the previous point we immediately obtain
\[
\varphi(z) \leq m(0)\beta^{3/4}\sqrt{z}.
\] (3.60)

Let \( z_n \to +\infty \) and let \( (\eta_n, \tilde{z}_n) \) be an admissible pair for \( \varphi(z_n) \) such that \( v_n \) is non-decreasing and
\[
\int_0^{\eta_n} (\alpha|v'_n| \wedge \beta) \, dx + \int_{\eta_n}^{\tilde{z}_n} |v''_n|^2 \, dx < \varphi(z_n) + 1.
\] (3.61)

Let \( \sigma_n \in (0, 1) \) be such that \( \int_{\{x \in I: |v'_n| \leq \beta/\alpha\}} |v'_n| \, dx = \sigma_n z_n \); since, by (3.60) and (3.61),
\[
m(0)\beta^{3/4}\sqrt{z_n} + 1 \geq \int_0^{\eta_n} (\alpha|v'_n| \wedge \beta) \, dx \geq \int_{\{x \in I: |v'_n| \leq \beta/\alpha\}} \alpha|v'_n| \, dx = \alpha \sigma_n z_n,
\]
it follows that \( \sigma_n \to 0 \). Consider the sets \( D_n := \{x \in I: v'_n > \beta/\alpha\} = \cup_{k=1}^{k_n} I_n^k \), where \( (I_n^k) \) is the collection of the connected components of \( D_n \). We denote also \( I_n^k := (a_n^k, b_n^k) \). Let \( \Phi \in C^2([0, 1]) \) be such that \( \Phi(0) = \Phi(0) = 0 \), \( \Phi(1) = 1 \), and \( \Phi'(1) = \beta/\alpha \) and, for every \( t \in [1, 1 + |D_n|] \) set
\[
i_n(t) := \min \left\{ k : \sum_{j=1}^k |I_n^j| \geq t - 1 \right\} \quad \tau_n(t) := t - 1 - \sum_{j=1}^{i_n(t)} |I_n^j|.
\]

We can now define the new sequence of admissible pairs \( (\tilde{\eta}_n, \tilde{v}_n) \) by \( \tilde{\eta}_n := |D_n| + 2 \) and \( \tilde{v}_n(t) = \int_0^t \tilde{v}_n^s(s) \, ds \), where
\[
\tilde{v}_n^s := \begin{cases} 
\Phi'(s) & \text{if } s \in [0, 1], \\
v_n \left( a_n^{i_n(s)} + \tau_n(s) \right) & \text{if } s \in [1, \tilde{\eta}_n - 1], \\
\Phi(\tilde{\eta}_n - s) & \text{if } s \in [\tilde{\eta}_n - 1, \tilde{\eta}_n].
\end{cases}
\]

Note that \( \tilde{v}_n \) is constructed by gluing together the pieces of \( v_n \) defined on the sets \( I_n^k \); since \( v_n(a_n^k) = v_n(b_n^k) = \beta/\alpha \) for every \( k \), we have \( \tilde{v}_n \in W^{2,2}(0, \tilde{\eta}_n) \). Therefore \( \tilde{v}_n \in S_{\tilde{\eta}_n, \tilde{z}_n} \) with \( \tilde{z}_n := (1 - \sigma_n)z_n + 2 \) and, since by construction
\[
\int_{D_n} (\alpha|v'_n| \wedge \beta) \, dx + \int_{D_n} |v''_n|^2 \, dx = \beta \tilde{\eta}_n + \int_0^{\tilde{\eta}_n} |\tilde{v}_n''|^2 \, dx - 2 \left( \int_0^1 |\Phi'| \, dx + \int_0^1 |\Phi''|^2 \, dx \right),
\]
recalling (3.61) and i) we can estimate
\[
\varphi(z_n) + 1 \geq \beta \tilde{\eta}_n + \int_0^{\tilde{\eta}_n} |\tilde{v}_n''|^2 \, dx - 2 \left( \int_0^1 |\Phi'| \, dx + \int_0^1 |\Phi''|^2 \, dx \right)
\]
\[
\geq \inf_{\eta > 0, \tilde{\eta}_n, \tilde{z}_n} \left( \beta \eta + \int_0^{\eta} |\tilde{v}_n''|^2 \, dx \right) - 2 \left( \int_0^1 |\Phi'| \, dx + \int_0^1 |\Phi''|^2 \, dx \right)
\]
\[
= m(0)\beta^{3/4}\sqrt{(1 - \sigma_n)z_n + 2} - 2 \left( \int_0^1 |\Phi'| \, dx + \int_0^1 |\Phi''|^2 \, dx \right),
\]
whence, taking into account that \( \sigma_n \to 0 \),
\[
\lim_{z \to +\infty} \frac{\varphi(z)}{m(0)\beta^{3/4}\sqrt{z}} \geq 1,
\]
which combined with (3.60), gives the thesis of the claim.
iii): For simplicity we treat in details only the case $\gamma = 1/2$. We take $b(t) := 1 + \log(1 + t)$ for $t > 0$ and $b(0) = 0$. Fix $p \in (0, 1)$ and take $(\eta, w)$ with $w \in S_{n, z}$ and satisfying
\[
\int_0^\eta |w'|^p \, dx + \int_0^\eta |w''|^2 \, dx = m(p)z^{\frac{p}{2-p}} \quad \text{and} \quad \eta \leq c(p)z^{\frac{p}{2-p}};
\]
this is possible by virtue of i) (see (3.59)). Then, since $b(t) \leq 1 + t^p$ we have
\[
\varphi(z) \leq (m(p) + c(p))z^{\frac{p}{2-p}}; \quad (3.62)
\]
since as $p$ varies in $(0, 1)$ the exponent $(2 - p)/(4 - p)$ varies in $(1/2, 1)$, from (3.62) we deduce that
\[
\lim_{z \to +\infty} \frac{\varphi(z)}{z^{1/2 + \varepsilon}} = 0 \quad \forall \varepsilon > 0.
\]
Now take two positive sequences $(\alpha_n)$ and $(\beta_n)$ with $\beta_n \to +\infty$ such that $b(t) \geq \beta_n(t) := \alpha_n \land \beta_n$, for every $t \geq 0$ and for every $n \in \mathbb{N}$. Calling $\varphi_n$ the function associated with $\beta_n$, by the claim proved above, we have
\[
\lim inf_{z \to +\infty} \frac{\varphi(z)}{\sqrt{z}} \geq \lim inf_{z \to +\infty} \frac{\varphi_n(z)}{\sqrt{z}} = m(0)\beta_n^{3/4},
\]
for every $n \in \mathbb{N}$; letting $n \to \infty$ we eventually complete the proof of (3.58). If $\gamma$ is any number in $(1/2, 1)$, take $b(t) = t^p \log(1 + t)$, where $p$ is such that $\gamma = (2 - p)/(4 - p)$, and argue as above. \hfill \Box

4 Some applications

In this subsection we are going to apply the results of the previous one to study the singular perturbations of the one-dimensional functionals of the form
\[
G_\varepsilon(u) = \frac{1}{\varepsilon} \int I f(\varepsilon^{1/q}|u'|) \, dx,
\]
where $q \geq 1$. More precisely, given a positive function $p(\varepsilon)$ such that $\lim_{\varepsilon \to 0^+} p(\varepsilon) = 0$, we set
\[
F_\varepsilon(u) = \begin{cases} \frac{1}{\varepsilon} \int I f(\varepsilon^{1/q}|u'|) \, dx + (p(\varepsilon))^3 \int |u''|^2 \, dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases} \quad (4.1)
\]
and we aim to classify all the possible $\Gamma$-limits generated by the family $(F_\varepsilon)$ depending on the asymptotic behaviour of the "rescaling" function $p$. Let us begin with the case $q > 1$. Let $f : [0, +\infty) \to [0, +\infty)$ be non-decreasing, continuous, and satisfying the following properties:

$\text{H1) } f$ is concave in $(x_1, +\infty)$ for some $x_1 > 0$;

$\text{H2) } \lim_{x \to +\infty} \frac{f(x)}{x^q} = \alpha > 0$, with $q > 1$;

$\text{H3) } \lim_{x \to +\infty} \frac{f(x)}{x} = 0$.

We will show that there exists a unique (up to asymptotic equivalence) rescaling function $r(\varepsilon)$ which generates non-trivial (i.e. non-zero) "free-discontinuity" functionals. Setting $h(x) := f(x)/x$, such a rescaling function is defined as:
\[
r(\varepsilon) := \frac{\varepsilon^{1/q}}{h \left( \frac{1}{\varepsilon^{1/q}} \right)}, \quad (4.2)
\]
where $q$ is the exponent appearing in $H2)$ while $q'$ denotes its Lebesgue conjugate exponent satisfying $1/q + 1/q' = 1$. 

Remark 4.1 Note that, for \( \varepsilon \) small enough, \( r \) is well defined, indeed, being \( f \) concave at infinity and sublinear, \( h \) becomes decreasing for \( x \) large enough. Moreover

\[
\lim_{n \to \infty} \frac{\sqrt{\varepsilon_n}}{r(\varepsilon_n)} = \lim_{n \to \infty} h^{-1}(\sqrt{\varepsilon_n}) = +\infty
\]

since \( h \downarrow 0 \) as \( x \to +\infty \).

Our main result is the following theorem.

**Theorem 4.2** Let \( I \subset \mathbb{R} \) be a bounded interval and let \( f : [0, +\infty) \to [0, +\infty) \) be a non-decreasing continuous function satisfying hypotheses H1), H2), and H3) and \( p(\varepsilon) \) be a positive function such that \( \lim_{\varepsilon \to 0^+} p(\varepsilon) = 0 \). Finally let \( (\varepsilon_n)_{n \in \mathbb{N}} \) be an infinitesimal sequence such that

\[
\lim_{n \to \infty} \frac{p(\varepsilon_n)}{r(\varepsilon_n)} = a > 0 \quad \text{and} \quad \exists \lim_{n \to +\infty} \frac{f\left(\frac{t \varepsilon_n^{1/q}}{r(\varepsilon_n)}\right)}{f\left(\frac{t \varepsilon_n^{1/q}}{r(\varepsilon_n)}\right)} =: b(t) \quad \forall t > 0. \tag{4.3}
\]

If \( b^0(1) = +\infty \) then the functionals \( F_{\varepsilon_n} \) (defined in (4.1)) \( \Gamma \)-converge with respect to the \( L^1 \)-metric to

\[
F(u) := \begin{cases} 
\alpha \int_I |u'|^q \, dx + \sum_{x \in S_u} \varphi^{(a)}(u^+(x) - u^-(x)) & \text{if } u \in SBV(I), \\
+\infty & \text{otherwise in } L^1(I); 
\end{cases} \tag{4.4}
\]

where \( \varphi^{(a)} \) is defined by (3.3) with \( b^{(a)}(t) := ab(t/a) \) instead of \( b(t) \).

Conversely, if \( b^0(1) = C < +\infty \), then \( \Gamma \)-lim\( n \to \infty \) \( F_{\varepsilon_n} = F \) with \( F \) given by

\[
F(u) := \begin{cases} 
\int_I g(|u'|) \, dx + \sum_{x \in S_u} \varphi^{(a)}(u^+(x) - u^-(x)) + C|D^c u| & \text{if } u \in BV(I), \\
+\infty & \text{if } x \in L^1(I) \setminus BV(I), 
\end{cases} \tag{4.5}
\]

where \( g := (\alpha x^a - Cx)^* \) while \( \varphi^{(a)} \) is as above. Moreover, in both cases, every sequence \( u_n \) such that \( \sup_n (F_n(u_n) + \|u_n\|_1) + \infty \) is strongly precompact in \( L^p \) for every \( p \geq 1 \).

An easy consequence of the theorem is the fact that, up to asymptotic equivalence, the function \( r \) defined in (4.2) is the unique nontrivial rescaling function; this is made precise by the following Corollary whose easy proof is left to the reader (see [2]).

**Corollary 4.3** Let \( I, f, \) and \( r \) be as in Theorem 3.2. Let \( (\varepsilon_n)_{n \in \mathbb{N}} \) and \( (a_n)_{n \in \mathbb{N}} \) be two sequences converging to \( 0 \) and, for every \( n \), set

\[
F_n(u) = \begin{cases} 
\frac{1}{\varepsilon_n} \int_I f(\varepsilon_n^{1/q}|u'|) \, dx + (a_n)^3 \int_I |u''|^2 \, dx & \text{if } u \in W^{2,2}(I), \\
+\infty & \text{otherwise in } L^1(I). 
\end{cases}
\]

If \( \lim_{n \to \infty} a_n/r(\varepsilon_n) = 0 \), then \( \Gamma \)-lim\( n \to \infty \) \( F_n = 0 \) with respect to the \( L^1 \)-metric; if \( \lim_{n \to \infty} a_n/r(\varepsilon_n) = +\infty \), then the functionals \( F_n \) \( \Gamma \)-converge to

\[
F(u) := \begin{cases} 
\alpha \int_I |u'|^q \, dx & \text{if } u \in W^{1,q}(I), \\
+\infty & \text{otherwise in } L^1(I), 
\end{cases}
\]
**Remark 4.4** Since $f$ is concave for $x$ large, from the previous remark and from the definition (4.3), it follows that $b(\cdot)$ is in turn concave. Moreover, again by the sublinearity and the concavity assumption we get the existence of $x_2 \geq x_1$ such that

$$f(a + b) \leq f(a) + f(b) \quad \forall a, b > x_2;$$

(4.6)

if $f$ is unbounded, we deduce

$$b(t) \leq \limsup_{z \to +\infty} \frac{f(tx)}{f(x)} \leq \limsup_{z \to +\infty} \frac{f([t] + 1)x)}{f(x)},$$

$$\leq \limsup_{z \to +\infty} \frac{([t] + 1)f(x)}{f(x)} = [t] + 1,$$

(4.7)

where $[t]$ denotes the integer part of $t$; if $f$ is bounded we get trivially $b(t) \equiv 1$. Finally, since $b(1) = 1$, taking into account the concavity it turns out that $b(t) > 0$ for any $t > 0$.

**Proof of Theorem 4.2.** Setting $r_n := p(\varepsilon_n)$ and $f_n(t) := \frac{1}{\varepsilon_n} f \left( \frac{\varepsilon_n}{1} \right)$, by $H2$ we get immediately that $f_n(t) \to \alpha t^q$ for every $t \geq 0$; moreover, using the identity $f \left( \frac{\varepsilon_n}{t} \right) = \frac{\varepsilon_n}{t} f(\varepsilon_n)$, which follows easily from the definition of $r$ (see (4.2)), for $t > 0$ it turns out

$$r_n f_n \left( \frac{t}{r_n} \right) = \frac{p(\varepsilon_n) r(\varepsilon_n)}{\varepsilon_n} \left( \frac{\frac{\varepsilon_n}{t} r(\varepsilon_n)}{\varepsilon_n} f \left( \frac{\varepsilon_n}{t} r(\varepsilon_n) \right) \frac{r(\varepsilon_n) t n}{\varepsilon_n} \right) = \frac{p(\varepsilon_n) f \left( \frac{\varepsilon_n}{t} r(\varepsilon_n) \right) n}{\varepsilon_n} \rightarrow \text{ab} \left( \frac{t}{a} \right) = b^{(a)}(t),$$

(4.8)

where we used (4.3). By the first part of Theorem 3.2 we therefore obtain the $\Gamma$-lim sup inequality. Concerning the other inequality, by Theorem 3.2 and Remark 3.15, it will be proved if for every $\delta > 0$ we are able to construct a family of functions $(f_n^{\delta})$ such that $f_n \geq f_n^{\delta}$, $f_n^{\delta}$ satisfies the structure condition st2 and finally

$$f_n^{\delta}(t) \to (1 - \delta)\alpha t^q \forall t \geq 0 \quad \text{and} \quad r_n f_n^{\delta} \left( \frac{t}{r_n} \right) \rightarrow b^{(a)}(t) \forall t > 0.$$

(4.9)

It is also clear that if we exhibit a function $f^{\delta}$ verifying

a) $f \geq f^{\delta}$,

b) $\lim_{t \to 0^+} \frac{f^{\delta}(t)}{t^q} = (1 - \delta)\alpha$,

c) $\lim_{t \to +\infty} \frac{f^{\delta}(t)}{f(t)} = 1$,

d) there exists $\mathfrak{r}$ such that $f^{\delta}$ is convex in $[0, \mathfrak{r}]$ and concave in $[\mathfrak{r}, +\infty)$,

then the family $f_n^{\delta}(t) := \frac{1}{\varepsilon_n} f^{\delta} \left( \frac{\varepsilon_n}{t} \right)$ enjoys all the required conditions. Therefore it remains only to construct such a $f^{\delta}$. By assumption we know that there exist $x', x''$ such that $f(t) \geq (1 - \delta)\alpha t^q$ for every $t \in [0, x']$ and $f$ is concave in $[x'', +\infty)$. Define $a(t) := (1 - \delta)\alpha \frac{x'}{x''}$, $g := \min\{(1 - \delta)\alpha t^q, a(t)\}$, and finally

$$f^{\delta}(t) := \begin{cases} g(t) & \text{if } t \leq x'', \\ f(t) + g(x'') - f(x'') & \text{if } t \geq x''. \end{cases}$$

it is easy to see that $f^{\delta}$ satisfies all conditions a),..., d) (see Figure 2). Finally the equicoerciveness of the family $(F_n)$ follows again from Theorem 3.2.

An easy consequence of Theorem 4.2 is the following compactness result.
Theorem 4.5 Let $I$, $f$, and $r$ as above and consider the family of functionals $F_\varepsilon$ defined in (4.1) with $p(\varepsilon)$ satisfying $0 < \liminf_{\varepsilon \to 0^+} \frac{p(\varepsilon)}{r(\varepsilon)} \leq \limsup_{\varepsilon \to 0^+} \frac{p(\varepsilon)}{r(\varepsilon)} < +\infty$. Then, for every infinitesimal sequence $(\varepsilon_n)_n$, there exist a subsequence, still denoted by $(\varepsilon_n)_n$ and a concave, non-decreasing function $b$ such that $\exists \Gamma$-lim$_n F_{\varepsilon_n} = F$ with respect to the $L^1$-convergence, where $F$ is either as in (4.4) or as in (4.5).

PROOF. It is sufficient to extract a subsequence such that (4.3) holds and then to apply Theorem 4.2. The existence of such a subsequence is an easy consequence of Helly’s Theorem.

□

Proposition 4.6 Let $f$ be a function satisfying the hypotheses H1), H2), and H3) of Theorem 4.2 and let us suppose in addition that

$$\exists \lim_{x \to +\infty} \frac{f(tx)}{f(x)} := b(t) \quad \forall t > 0. \quad (4.10)$$

Let $F_\varepsilon$ the functional defined in (4.1), with $p$ satisfying $\lim_{\varepsilon \to 0^+} \frac{p(\varepsilon)}{r(\varepsilon)} = a > 0$, where $r$ is the rescaling function defined in (4.2). If $b^0(1) = +\infty$, then $\Gamma$-$\lim_{\varepsilon \to 0^+} F_\varepsilon = F$ with respect to the $L^1$-metric, with $F$ given by

$$F(u) := \begin{cases} \int_I |u'|^q \, dx + m(\gamma) a^{\frac{3(q+\gamma)}{q-1^+}} \sum_{x \in S_n} (u^+ - u^-)^{\frac{2+\gamma}{q-1^+}} & \text{if } u \in SBV(I) \\ +\infty & \text{in } L^1(I) \setminus SBV(I), \end{cases}$$

where $\gamma = \log b(\varepsilon)$ and $m(\gamma)$ is the constant defined in (3.56). If $b^0(1) < +\infty$ the family $F_\varepsilon$ $\Gamma$-converges to the functional $F$ given by

$$F(u) := \begin{cases} \int_I g_\varepsilon(|u'|) \, dx + |D^*u| & \text{if } u \in BV(I) \\ +\infty & \text{in } L^1(I) \setminus BV(I), \end{cases}$$

where $g_\varepsilon = (\alpha x^2 \wedge x)^{**}$. 

Figure 2: The construction of $f^\delta$. 

PROOF. From Theorem 4.2, it is clear that \( \Gamma \)-\( \lim_{\varepsilon \to 0^+} F_\varepsilon = F \) where \( F \) is the functional defined either in (4.4) or in (4.5). It remains only to prove that
\[
\varphi^{(a)}(z) = m(\gamma)a \frac{3^{1-\gamma}}{4\gamma - 3}\gamma z^{\frac{2+\gamma}{4\gamma - 3}} \quad \forall z > 0, \tag{4.11}
\]
if (4.3) hold true, or
\[
\varphi^{(a)}(z) = z \quad \forall z > 0, \tag{4.12}
\]
otherwise. First of all note that from (4.10) it follows immediately that \( b(st) = b(s)b(t) \) for \( t, s > 0 \) and therefore \( b(t) = t^\gamma \) for \( t > 0 \), with \( \gamma = \log b(e) \); by remark 4.4 (and in particular by (4.7)), we have that \( \gamma \in [0, 1] \). If \( \gamma < 1 \), (4.11) follows from Proposition 3.18 since \( b^{(a)}(t)ab(t/a) = a^1 \gamma t^\gamma \). If \( \gamma = 1 \), then by Lemma 3.5, we get (4.12). \( \square \)

Let us see now some examples. We will use the following notation: given two functions \( r_1 \) and \( r_2 \) we will write \( r_1 \simeq r_2 \) if they are asymptotically equivalent, that is if \( \lim_{\varepsilon \to 0^+} \frac{r_1(\varepsilon)}{r_2(\varepsilon)} = 1 \).

**Example 4.7** Let \( \gamma \) belong to \( [0, 1] \) and set \( f(x) = \frac{\alpha x^2}{\varepsilon^{\gamma} - 1} \); using the definitions (see (4.2) and (4.3)), it is easy to see that \( r(\varepsilon) \simeq \varepsilon^{\frac{2-\gamma}{2\gamma}} \) and \( b(t) = t^\gamma \), and therefore, setting
\[
F_\varepsilon(u) := \begin{cases} 
\int_I \frac{\alpha}{1 + \frac{\varepsilon}{2\gamma - 3}} |u'|^2 \, dx + a^3 \frac{2-\gamma}{2\gamma} \int_I |u''|^2 \, dx & \text{if } u \in W^{2,2}(I), \\
+\infty & \text{otherwise in } L^1(I),
\end{cases}
\]
by Proposition 4.6, we have that the functionals \( F_\varepsilon \) \( \Gamma \)-converge to
\[
F^\gamma(u) := \begin{cases} 
\alpha \int_I |u'|^2 \, dx + m(\gamma)a \frac{3^{1-\gamma}}{4\gamma - 3}\gamma \sum_{\varepsilon \in S_\varepsilon} (u^+ - u^-)^{\frac{2+\gamma}{4\gamma - 3}} & \text{if } u \in SBV(I), \\
+\infty & \text{in } L^1(I) \setminus SBV(I),
\end{cases}
\tag{4.13}
\]
as \( \varepsilon \to 0^+ \), with respect to the \( L^1 \)-metric, so that we recover the result of Dubs, Bouchitté & Seppecher (see [9]). Note that as \( \gamma \) varies in \( [0, 1] \), the exponent \( \frac{2+\gamma}{4\gamma} \) varies in \( \left[\frac{1}{2}, 1\right] \). Moreover, note that \( m(0) \) can be easily computed and it is equal to \( 2\sqrt{3}/2 + \sqrt{2}/3 \) (see [2]).

**Example 4.8** Let \( f(x) = (1 + x^\gamma) \log(1 + \alpha x^2) \) with \( \gamma \in [0, 1] \). We show now that
\[
r(\varepsilon) \simeq (1 - \gamma)^{\frac{1}{2^\gamma - 3}} \frac{\varepsilon^{\frac{2-\gamma}{2\gamma}}}{(\log \frac{1}{\varepsilon})^{\frac{2-\gamma}{2\gamma}}}.
\tag{4.14}
\]
Indeed, with the same notations of Theorem 3.2, we have
\[
\lim_{\varepsilon \to 0^+} \frac{r(\varepsilon)(\log \varepsilon)^{\frac{1}{2^\gamma - 3}}}{\varepsilon^{\frac{2-\gamma}{2\gamma}}} = \lim_{\varepsilon \to 0^+} \frac{\sqrt{\varepsilon}}{h(1/\varepsilon)} (\log \frac{1}{\varepsilon})^{\frac{1}{2^\gamma - 3}} = \lim_{y \to +\infty} \frac{(\log \frac{1}{y})^{\frac{1}{2^\gamma - 3}}}{yh^{\frac{1}{2^\gamma - 3}}} = (1 - \gamma)^{\frac{1}{2^\gamma - 3}},
\]
where we performed the change of variable \( y = h \frac{1}{\sqrt{\varepsilon}} \).

We finally observe that \( b(t) = \lim_{t \to +\infty} \frac{f(tx)}{f(x)} = t^\gamma \) for all \( t > 0 \); therefore, setting
\[
F^\gamma(u) := \begin{cases} 
\frac{1}{\varepsilon} \int_I (1 + \varepsilon^\gamma |u'|^2) \log(1 + \alpha^2 |u'|^2) \, dx \\
+\infty & \text{if } u \in W^{2,2}(I),
\end{cases}
\]
otherwise in \( L^1(I) \),
by (4.14) and by Proposition 4.6 we obtain that the sequence $F^\varepsilon_{\alpha}$ $\Gamma$-converges, as $\varepsilon \to 0^+$, to the functional $F^\gamma$ defined in (4.13). In particular, taking $\gamma = 0$, we prove that the singular perturbations of the rescaled Perona-Malik functionals

$$F_{\alpha}(u) := \begin{cases} \frac{1}{\varepsilon} \int_I \log(1 + \varepsilon \alpha |u'|^2) \, dx \quad & \text{if } u \in W^{2,2}(I), \\ \frac{a}{(\log \frac{1}{\varepsilon})^3} \int_I |u''|^2 \, dx \quad & \text{otherwise in } L^1(I), \end{cases}$$

$\Gamma$-converge to $F^0$, as announced in the Introduction.

**Remark 4.9** Let $f_1$ and $f_2$ be two functions satisfying the hypotheses of Theorem 3.2 and let $r_1$ and $r_2$ be the rescaling functions associated with $f_1$ and $f_2$ respectively according to (4.2) and, for $\varepsilon > 0$ and $i = 1, 2$ denote by $F_{i,\varepsilon}$ the functional

$$F_{i,\varepsilon}(u) = \begin{cases} \frac{1}{\varepsilon} \int_I f_i(\sqrt{\varepsilon} |u'|) \, dx + (r_i(\varepsilon))^3 \int_I u''^2 \, dx \quad & \text{if } u \in W^{2,2}(I), \\ +\infty \quad & \text{otherwise in } L^1(I). \end{cases}$$

Suppose in addition that

$$\lim_{\varepsilon \to 0^+} \frac{f_1(x) \log^2(x)}{f_2(x)} = 1; \quad \text{(4.15)}$$

then, for every infinitesimal sequence $(\varepsilon_n)_{n}$, $\Gamma$-lim$_{n \to \infty} F_{1,\varepsilon_n} = F \iff \Gamma$-lim$_{n \to \infty} F_{2,\varepsilon_n} = F$; in other words, functions asymptotically differing by a logarithmic factor generate the same $\Gamma$-limits. To prove this fact we pass to a subsequence such that

$$\exists \lim_{n \to +\infty} \frac{f_1(t \sqrt{\varepsilon_n})}{f_1(t \sqrt{r_1(\varepsilon_n)})} =: b_1(t) \quad \forall t > 0 \quad \text{and} \quad \exists \lim_{n \to +\infty} \frac{f_2(t \sqrt{\varepsilon_n})}{f_2(t \sqrt{r_2(\varepsilon_n)})} =: b_2(t) \quad \forall t > 0,$$

and we observe that, by virtue of (4.15), we have $b_1 \equiv b_2$; we conclude by applying Theorem 4.2. Note that the results of Example 4.8 can be derived from Examples 4.7, using the present remark.

**Example 4.10** Let $f_\alpha(x) := \frac{\alpha x^2}{(1 + \varepsilon \log(\varepsilon + 1))^2}$; then, by easy computations, the rescaling function $r$ defined in (4.2) satisfies $r(\varepsilon) \sim \frac{\varepsilon}{\varepsilon + x}$. Moreover $b(t) = \lim_{t \to +\infty} \frac{t \sqrt{r(t)}}{t \sqrt{r(t)}} = t$ for every $t > 0$; therefore, setting

$$F_{\alpha,\varepsilon}(u) := \begin{cases} \frac{1}{\varepsilon} \int_I f_\alpha(\sqrt{\varepsilon} |u'|) \, dx + \left(\frac{\varepsilon}{\varepsilon + x}\right)^3 \int_I |u''|^2 \, dx \quad & \text{if } u \in W^{2,2}(I), \\ +\infty \quad & \text{otherwise in } L^1(I), \end{cases}$$

by Proposition 4.6 we obtain that the family $F_{\varepsilon}$ $\Gamma$-converges, with respect to $L^1$-metric, to

$$F_\alpha(u) := \begin{cases} \int_I g_\alpha(|u'|) \, dx \quad & \text{if } u \in BV(I) \\ +\infty \quad & \text{in } L^1(I) \setminus BV(I), \end{cases}$$

where $g_\alpha = (\alpha x^2 \wedge x)''$. Note that $g_\alpha(x) \uparrow x$ as $\alpha \to +\infty$, and therefore

$$\Gamma$-lim_{\alpha \to +\infty} \Gamma$-lim_{\varepsilon \to 0^+} F_{\alpha,\varepsilon} = \Gamma$-lim_{\alpha \to +\infty} F_\alpha = G,$$

with $G$ given by

$$G(u) := \begin{cases} Du \quad & \text{if } u \in BV(I), \\ +\infty \quad & \text{otherwise in } L^1(I). \end{cases}$$
**Remark 4.11** The hypothesis $H3)$ is in some sense necessary; indeed suppose that $f$ is an increasing function satisfying $H1), H2),$ and \( \lim_{\varepsilon \to +\infty} \frac{f(\varepsilon)}{\varepsilon} = C > 0. \) Then it is easy to see that the functionals

\[
G_{\varepsilon}(u) := \begin{cases} 
\frac{1}{\varepsilon} \int_{I} f(\sqrt{\varepsilon}|u'|) \, dx, & \text{if } u \in C^1(I), \\
+\infty & \text{otherwise in } L^1(I),
\end{cases}
\]

\( \Gamma \)-converge in the $L^1$-topology, to the functional

\[
G(u) := \begin{cases} 
\alpha \int_{I} |u'|^q \, dx, & \text{if } u \in W^{1,q}(I), \\
+\infty & \text{otherwise in } L^1(I),
\end{cases}
\]
as $\varepsilon \to 0^+$. We leave the details to the reader.

Let us see what happens when in (4.1) the function $f$ has a finite strictly positive derivative at the origin so that $q = 1$.

**Theorem 4.12** Let $f : [0, +\infty) \to [0, +\infty)$ be continuous, non-decreasing, differentiable in $0$ with $f'(0) > 0$, and concave in $(x_1, +\infty)$ for a suitable $x_1 > 0$. Then the family

\[
F_{\varepsilon}(u) := \begin{cases} 
\frac{1}{\varepsilon} \int_{I} f(\varepsilon|u'|) \, dx + \varepsilon^3 \int_{I} |u''|^2 \, dx & \text{if } u \in W^{2,2}(I), \\
+\infty & \text{otherwise in } L^1(I),
\end{cases}
\]
\( \Gamma \)-converges with respect to the $L^1$-norm to the functional

\[
F(u) := \begin{cases} 
f'(0) \int_{I} |u'| \, dx + \sum_{s_a} \varphi(u^+ - u^-) + f'(0) D^2 u | & \text{if } u \in BV(I), \\
+\infty & \text{otherwise in } L^1(I),
\end{cases}
\]

where $\varphi$ is the function defined in (3.3) with $b = f$. Moreover every sequence $u_{\varepsilon}$ such that \( \sup_{\varepsilon} (F_{\varepsilon}(u_{\varepsilon}) + ||u_{\varepsilon}||_1) < +\infty \) is strongly precompact in $L^p$ for every $p \geq 1$.

**Proof.** Take an infinitesimal sequence $(\varepsilon_n)$ and consider the family of functions $f_n := (1/\varepsilon_n)f(\varepsilon_n \cdot)$: we clearly have that $f_n(t) \to f'(0)t$ for every $t > 0$ and $\varepsilon_n f_n(\varepsilon_n t) = f(t)$ for every $t > 0$ and every $n \in \mathbb{N}$, so that $(f_n)$ verifies (3.5) and (3.6) with $g = f'(0)t$ and $b = f$. Now construct a sequence of functions $(f_{\varepsilon})$ such that $f \geq f_{\varepsilon}$ for every $k$, $(f_{\varepsilon})(0)$ as $k \to +\infty$, and $f_{\varepsilon}$ is liminarily in $[0, y_k]$ and concave in $[y_k, +\infty)$ for a suitable $y_k > 0$ (it is clear that under our assumptions such a construction is possible); then, setting $f_{\varepsilon}^0(t) := (1/\varepsilon_n)f_{\varepsilon}(\varepsilon_n t)$, we have that the family $(f_{\varepsilon})_{n,k}$ satisfies the weaker structure assumption introduced in Remark 3.15. At this point we can conclude by applying Theorem 3.2. \( \square \)

The following example is in the spirit of Theorem 3.17.

**Example 4.13** Given a convex non-decreasing positive function $g$ and a concave positive function $b$ satisfying $b^0(1) = g^\infty(1) = C \in (0, +\infty)$, we have that the family

\[
F_{\varepsilon}(u) := \begin{cases} 
\int_{I} \left[ g(|u'|) + \left( \frac{1}{\varepsilon} b(\varepsilon|u'|) + g(0) \right) \right] \, dx + \varepsilon^3 \int_{I} |u''|^2 \, dx & \text{if } u \in W^{2,2}(I), \\
+\infty & \text{otherwise}
\end{cases}
\]
\( \Gamma \)-converges to the functional $F_{0,g}$ defined in (3.4) which, under our assumptions, takes the form

\[
\int_{I} g(|u'|) \, dx + \sum_{s_a} \varphi(u^+ - u^-) + C|D^2 u| & \text{if } u \in BV(I),
\]
where \( \varphi \) is the function associated with \( b \) according to (3.3). It is enough to apply Theorem 3.2 to the family 
\[
 f_\varepsilon := g(t) \land (\frac{1}{\varepsilon} b(t) + g(0)), \quad \text{after noting that}
\]
\[
f_\varepsilon (t) \rightarrow g(t) \land (b^0(t) + g(0)) = g(t) \quad \text{and} \quad \varepsilon f_\varepsilon \left( \frac{t}{\varepsilon} \right) \rightarrow b(t) \land g^\infty (t) = b(t).
\]

5  The \( N \)-dimensional case

In the section we aim to extend the results of the previous ones to the \( N \)-dimensional case. Let us fix first some notations: for \( u \in W^{2,2}(\Omega) \), we denote its hessian matrix by \( \nabla^2 u \) and, given a square matrix \( A \) we consider the norm defined by 
\[
\| A \| := \sup_{|\xi| = 1} A \xi \cdot \xi.
\]

It is convenient to introduce the following definition.

**Definition 5.1** Given \( X \subseteq L^1(\Omega) \) we say that the sequence of functionals \( F_n : X \rightarrow \mathbb{R} \cup \{+\infty\} \) steadily \( \Gamma \)-converges in \( X \) to \( F : X \rightarrow \mathbb{R} \cup \{+\infty\} \) (and we will write \( \Gamma^s \)-lim\( \nrightarrow -\infty \), \( F_n = F \) or, shortly, \( F_n \rightharpoonup \Gamma \rightarrow F \)) if, for every \( p \geq 1 \), \( F_n \rightharpoonup \Gamma \rightarrow F \) -converges to \( F \rightharpoonup \Gamma \rightarrow F \) with respect to the \( L^p \)-convergence. Equivalently we have that \( \Gamma^s \)-lim\( \nrightarrow -\infty \) \( F_n = F \) if and only if the two following conditions are satisfied:

i) for every \( (u_n)_n \subset X \) such that \( u_n \rightarrow u \in X \) in \( L^1 \), we have
\[
\liminf_{n \rightarrow \infty} F_n (u_n) \geq F(u);
\]

ii) for every \( u \in X \cap L^p(\Omega) \), there exists a sequence \( (u_n)_n \subset X \cap L^p(\Omega) \) such that
\[
u_n \rightarrow u \text{ in } L^p \quad \text{and} \quad \limsup_{n \rightarrow \infty} F_n (u_n) \leq F(u).
\]

We will also say that \( G \) is the steady relaxed functional of \( F \) if \( G \) is the \( \Gamma^s \)-limit of the constant sequence \( F_n = F \).

We underline that, thanks to Remark 3.9, in the one dimensional case we have in fact proved the steady \( \Gamma \)-convergence in the whole \( L^1(I) \) of the functionals \( F_n \). The main result of the section is the following theorem.

**Theorem 5.2** Let \( \Omega \subset \mathbb{R}^N \) be an open bounded set with Lipschitz boundary and let \( f_n, r_n \) satisfy hypotheses i), ii), and iii) of Theorem 3.2. For every \( n \in \mathbb{N} \), consider the following \( N \)-dimensional version of the functional \( F_n \) defined in (3.1):
\[
F^n (u) = \begin{cases}
\int_\Omega f_n (|\nabla u|) dx + (r_n)^3 \int_\Omega \| \nabla^2 u \|^2 dx & \text{if } u \in W^{2,2}(\Omega), \\
+\infty & \text{otherwise in } L^1(\Omega).
\end{cases}
\]

Then
\[
\Gamma \text{-lim inf}_{n \rightarrow \infty} F^n \geq F^N_{b, g},
\]
with respect to the \( L^1 \)-convergence, where \( F^N_{b, g} \) is the \( N \)-dimensional version of \( F_{b, g} \) given by
\[
F^N_{b, g}(u) := \begin{cases}
\int_\Omega g_1 (|\nabla u|) dx + \int_{S_u} \varphi_1 (u^+ (x) - u^- (x)) dH^N \ + (g^\infty (1) \land b^0 (1)) |D^e u| & \text{if } u \in GBV(\Omega), \\
+\infty & \text{otherwise.}
\end{cases}
\]

Suppose now that, for every \( n \), \( f_n \) satisfies the following additional growth conditions:
\textbf{gr1}) there exists $C$, $C_0 > 0$ and $q \geq 1$ such that $f_n(t) \leq C(1 + t^q)$ and $C_0 t^q \leq g(t) \leq C(1 + t^q)$ for every $t \geq 0$.

\textbf{gr2}) for every $\alpha > 0$ there exists $c(\alpha) > 0$ such that $f_n(at) \leq c(\alpha) f_n(t)$ for every $t \geq 0$, where $C$, $C_0$, $q$, and $c(\alpha)$ are independent of $n$. Then the sequence $F_n^N$ $\Gamma$-converges in $GSBV^q$ to $F_{b,g}^N$. Moreover if $g^\infty(1) \wedge b^0(1) < +\infty$, actually we have that $\Gamma$-lim$n \to \infty$ $F_n^N = F_{b,g}^N$ in the whole $L^1(\Omega)$.

\textbf{Remark 5.3} Note that, for technical reasons, when $g^\infty(1) \wedge b^0(1) = +\infty$ in the $N$-dimensional case we are able to represent the $\Gamma$-limit only on $GSBV^q(\Omega)$; to complete the result one would have to show that every $u \in GSBV(\Omega)$ can be approximated in $L^1$ by a sequence $u_j \in GSBV^q(\Omega)$ such that $F_{b,g}^N(u_j) \to F_{b,g}^N(u)$ (see the introduction).

\textbf{Remark 5.4} Note that if $g$ satisfies both \textbf{gr1}) and \textbf{gr2}), also the family $f_\varepsilon$ constructed in Proposition 3.17 verifies the same growth conditions.

\textbf{Proof.} Let us prove (5.2). The inequality will be proved by means of the so called slicing method, which relies on the use of Theorem 2.1.

Let us suppose for simplicity that $g^\infty(1) \wedge b^0(1) = +\infty$; the other case can be treated in an analogous way. First of all we observe that, for $\xi \in S^n^1$, for $u \in W^{2,2}(\Omega)$, and for $A \in A(\Omega)$ we have, by Fubini’s Theorem and by the monotonicity of $f_n$,

$$F_n^N(u, A) = \int_{\Pi_{\xi}} \int_{A^R_{\xi}} f_n((\nabla u(y + t\xi)) + (r_n)^3 \| \nabla u(y + t\xi) \|^2) \, dt \, dH^N \, 1(y)$$

$$\geq \int_{\Pi_{\xi}} \int_{A^R_{\xi}} f_n((|u^y_{\xi}'|)) + (r_n)^3 (|u^y_{\xi}|)^2) \, dt \, dH^N \, 1(y)$$

$$= \int_{\Pi_{\xi}} F_n(u^y_{\xi}, A^y_{\xi}) \, dH^N \, 1(y),$$

where, $\Pi_{\xi}$ is the hyperplane orthogonal to $\xi$ while $A^y_{\xi}$ and $u^y_{\xi}$ are the one dimensional sections defined in Subsection 2.1. Let $u_n \to u$ in $L^1(A)$ be such that $\sup_n F_n^N < +\infty$ and note that, for every $\xi \in S^n^1$ and for almost every $y \in A^R_{\xi}$, $(u^y_{\xi})_n \to (u^y_{\xi})$ in $L^1(A^y_{\xi})$, hence, recalling that, by our assumptions, $F_n(\cdot, A^y_{\xi}) \xrightarrow{\text{loc}} F_{b,g}(\cdot, A^y_{\xi})$ and using Fatou’s Lemma,

$$\liminf_{n \to \infty} F_n^N(u_n, A) \geq \int_{\Pi_{\xi}} \liminf_{n \to \infty} F_n((u^y_{\xi})_n, A^y_{\xi}) \, dH^N \, 1(y)$$

$$\geq \int_{\Pi_{\xi}} \left( \int_{A^R_{\xi}} g((u^y_{\xi}')) + \sum_{(s_u)_{\xi} \cap A^y_{\xi}} \varphi((u^y_{\xi})^+ - (u^y_{\xi}^-)) \right) \chi_{\Pi_{\xi}} \, dH^N \, 1(y).$$

From (5.3), by virtue of Theorem 2.1, we get $u \in GSBV(\Omega)$ and

$$\Gamma \liminf_{n \to \infty} F_n^N(u, A) \geq \alpha \int_A g(|\nabla u \cdot \xi|) \, dx + \int_{S_u \cap A} \varphi(u^+ - u) |\nu_u \cdot \xi| \, dH^N \, 1 = \int_A \psi_\xi(x) \lambda,$$

where we have set

$$\lambda := L^N + \varphi(u^+ - u) \chi_{S_u},$$

and

$$\psi_\xi := g(|\nabla u \cdot \xi|) \chi_{S_u} + |\nu_u \cdot \xi| \chi_{S_u};$$

since (5.4) holds true for every $\xi$ and for every $A \in A(\Omega)$, choosing a sequence $(\xi_k)_{k \in \mathbb{N}}$ dense in $S^n^1$, and by applying Lemma 3.13 (with $\nu(\cdot) := \Gamma \liminf_{n \to \infty} F_n^N(u, \cdot)$), we finally obtain

$$\Gamma \liminf_{n \to \infty} F_n^N(u) \geq \int_{\Omega} \sup \psi_\xi \, dx = \int_{\Omega} g(|\nabla u|) \, dx + \int_{S_u} \varphi(u^+ - u) \, dH^N \, 1,$$
as desired.

Concerning the $\Gamma$-lim sup inequality, we adapt the proof given in [3]. In the sequel we will assume that gr1 and gr2 hold true. For every $p \geq 1$ we denote by $G_p : L^p(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ the following functional

$$
G_p(u, A) := \inf_{n \to \infty} \{ \text{sup}_{n} F_n^N (u_n, A) : u_n \to u \text{ in } L^p(A) \};
$$

our thesis is then equivalent to prove that $G_p(u, \Omega) \leq F_{b, g}^N (u, \Omega)$ for every $u \in GSBV^q(\Omega) \cap L^p(\Omega)$. It is clear that

$$
G_{p_1}(u, A) \leq G_{p_2}(u, A),
$$

for every $1 \leq p_1 < p_2$, for every $u \in L^{p_2}(\Omega)$, and for every $A \in \mathcal{A}(\Omega)$.

Step 1. Let $\Pi$ be an affine hyperplane, and denote by $\Pi^+$ and $\Pi^-$ the two open half-spaces whose union gives $\mathbb{R}^N \setminus \Pi$ and by $\nu$ the unit normal vector to $\Pi$ which points towards $\Pi^+$. Then, for every $A \in \mathcal{A}(\Omega)$ and for every $z \in \mathbb{R}$,

$$
G_p(z \chi_{\Pi^+}, A) \leq \varphi(|z|) \mathcal{H}^N 1(\Pi \cap A) = F_{b, g}^N (z \chi_{\Pi^+}, A) \quad \forall p \geq 1.
$$

First of all, since

$$
\lim_{t \to 0} \mathcal{H}^N 1(\{x \in A : d(x) = t\}) = \mathcal{H}^N 1(\Pi \cap A),
$$

for $\delta \in (0, 1)$, we can choose $\eta > 0$ such that

$$
\sup_{t \in \{ \eta, \eta \}} \mathcal{H}^N 1(\{x \in A : d(x) = t\}) \leq (1 + \delta) \mathcal{H}^N 1(\Pi \cap A).
$$

Let $u_n \to z \chi_{(0, +\infty)}$ be the one dimensional recovery sequence constructed in the previous section which satisfies $\|u_n\|_\infty \leq \varepsilon$, $u_n \equiv z \chi_{(0, +\infty)}$ in $\mathbb{R} \setminus (-\eta, \eta)$, and

$$
\lim_{n \to \infty} F_n(u_n, (-\eta, \eta)) = F_{b, g}(z \chi_{(0, +\infty)}, (-\eta, \eta)) = \varphi(|z|);
$$

we recall also that, for a suitable $K > 0$,

$$
r_n ||u_n'\|_\infty \leq K.\quad (5.8)
$$

We define, for every $x \in \Omega$, $v_n(x) := u_n(d(x))$, where $d$ is the signed distance function from $\Pi$, positive in $\Pi^+$ and negative in $\Pi^-$, clearly $v_n \in W^{2,2}(\Omega)$ and $v_n \to z \chi_{\Pi^+}$ in $L^p(\Omega)$ for every $p \geq 1$; moreover, using co-area formula (see (2.1)), (5.6), and (5.8), we can estimate

$$
F_n^N (v_n, A) = \int_{\Omega} F_n (|u'(d)|) \, dx + (r_n)^3 \int_{\Omega} (\|u_n'(d)\| \nabla d \otimes \nabla d + u_n'(d) \nabla^2 d)^2 \, dx
$$

$$
\leq \int_{\Omega} \int_{\{x : d(x) = t\}} f_n (|u_n'(t)|) \, d\mathcal{H}^N 1 \, dt
$$

$$
+ \int_{\Omega} \int_{\{x : d(x) = t\}} ((1 + \varepsilon) \|u_n''(t)\|^2 + c_r \|u_n'(t)\|^2) \|\nabla^2 d\| \, d\mathcal{H}^N 1 \, dt
$$

$$
\leq \int_{\Omega} \int_{\{x : d(x) = t\}} \mathcal{H}^N 1(\{x \in A : d(x) = t\}) \, dt
$$

$$
+ c_r K r_n \|\nabla^2 d\|_\infty \int_{\Omega} \mathcal{H}^N 1(\{x \in A : d(x) = t\}) \, dt
$$

$$
\leq (1 + \varepsilon) (1 + \delta) \mathcal{H}^N 1(\Pi \cap A) F_n(u_n, (-\eta, \eta)) + c_r (1 + \delta) \mathcal{H}^N 1(\Pi \cap A) 2K \eta r_n \|\nabla^2 d\|_\infty,
$$

where we denoted by $(\Pi)_\eta$ the $\eta$-neighbourhood of $\Pi$. From the last inequality, taking into account (5.7), we deduce

$$
\limsup_{n \to \infty} F_n^N (v_n, A) \leq (1 + \varepsilon)(1 + \delta) \mathcal{H}^N 1(\Pi \cap A) \varphi(|z|);
$$
since $\delta$ and $\varepsilon$ are arbitrary, STEP 1 is proved.

STEP 2. Let $u = \sum_{i=1}^{k} z_i \chi_{E_i}$ with $E_i$ closed polyhedra such that $\hat{E}_i \cap \hat{E}_j = \emptyset$ for $i \neq j$. Then

$$G_p(u, A) \leq F^N_{b,d}(u, A),$$

for all $A \in A(\Omega)$ and for every $p \geq 1$.

The proof is based on a standard partition of unity argument, we refer to Proposition 2.6 of [3] for the
details.

STEP 3. Let $A', A, B \in A(I)$ such that $A' \subset A$ and let $\phi$ be a cut-off function between $A'$ and $A$. Then

there exists a positive constant $C > 0$ such that, for every $u, v \in W^{2,2}(\Omega) \cap L^q(\Omega)$, we have

$$F^N_n(\phi u + (1 - \phi)v, A' \cup B) \leq F^N_n(u, A) + F^N_n(v, B) + \mathcal{C}(F^N_n(u, A) + F^N_n(v, B)) + C\mathcal{L}^N_S(S)$$

$$+ C(r_n)^3 (|\nabla \phi|^2_2 \|u - v\|^2 + \|\nabla^2 u - \nabla^2 v\|^2)$$

$$\leq F^N_n(u, A) + F^N_n(v, B) + \int_S f_n(3(u - v)\nabla \phi) + f_n(3|\nabla u|) + f_n(3(1 - \phi)\nabla v)) \|u - v\|^2 + \|\nabla^2 u\|^2 + \|\nabla^2 v\|^2$$

using (5.9) we can continue our estimate

$$(*) \leq F^N_n(u, A) + F^N_n(v, B) + (C_1 + c(3))(F^N_n(u, S) + F^N_n(v, S)) + \int_S f_n(3(u - v)\nabla \phi) \|u - v\|^2$$

recalling (GR1), from the last inequality we easily get (5.9).

STEP 4. Let $A', A, B, S$ be as in STEP 3 and $p \geq q$. Then for every $u, v \in L^p(\Omega)$ and for every $K \in \mathbb{N}$ there exists a cut-off function $\phi_K$ between $A$ and $A'$, such that

$$G_p(\phi_K u + (1 - \phi_K)v, A' \cup B) \leq \left(1 + \frac{C}{K}\right) (G_p(u, A) + G_p(v, B)) + C\mathcal{L}^N_S(S),$$

where $d := \text{dist}(A', \Omega \setminus A)$.

First of all choose $u_n, v_n \in W^{2,2}(\Omega)$ such that $u_n \rightarrow u, v_n \rightarrow v$ in $L^p(\Omega)$ and

$$G_p(u, A) = \lim_{n \rightarrow \infty} F^N_n(u_n, A) \quad \text{and} \quad G_p(u, B) = \lim_{n \rightarrow \infty} F^N_n(v_n, B).$$

For $j \in \{0, 1, \ldots, K\}$ we consider the set

$$A^j = \left\{ x \in A : \text{dist}(x, A') < j \frac{d}{K} \right\}$$

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for any $j \in \{0, 1, \ldots, K - 1\}$ we choose a cut-off function $\phi^K_j$ between $A^K_j$ and $A^K_{j+1}$ such that
\[
\|\nabla \phi^K_j\|_\infty \leq \frac{2K}{d};
\] (5.11)
finally we set $S^K_j := (A^K_{j+1} \setminus \overline{A^K_j}) \cap B$. By using (5.9) and (5.11), we get
\[
F_n^N(\phi^K_j u_n + (1 - \phi^K_j) v_n, A' \cup B)
\leq \ F_n^N(u_n, A) + F_n^N(v_n, B) + C(F_n^N(u_n, S^K_j) + F_n^N(v_n, S^K_j) + \frac{K}{d} \|u - v\|^2_{L^2(S^K_j)}
+ C(r_n)^3(\|\nabla \phi^K_j\|^2 \|\nabla u_n - \nabla v_n\|^2_{L^2(S^K_j)} + \|\nabla \phi^K_j\|^2 \|u_n - v_n\|^2_{L^2(S^K_j)}).
\] (5.12)
Passing to a subsequence, if needed, it follows that there exists $j_K \in \{0, 1, \ldots, K - 1\}$ such that
\[
F_n^N(\phi^K_{j_K} u_n + (1 - \phi^K_{j_K}) v_n, A' \cup B) \\leq \frac{1}{K} \sum_{j=0}^{K-1} F_n^N(\phi^K_j u_n + (1 - \phi^K_j) v_n, A' \cup B)
\] \leq \left(1 + \frac{C}{K}\right)(F_n^N(u_n, A) + F_n^N(v_n, B)) + C(r_n)^3(\|\nabla u_n - \nabla v_n\|^2_{L^2(S)} + \|u_n - v_n\|^2_{L^2(S)}),
\] (5.12)
for every $n \in \mathbb{N}$. Recall that by Nirenberg inequality (see [25]), there exists $M > 0$ such that for all $u \in W^{2,2}(S)$
\[
\|\nabla u\|_{L^2(S)} \leq M(\|\nabla^2 u\|^{1/2}_{L^2(S)} \|u\|^{1/2}_{L^2(S)} + \|u\|_{L^2(S)}),
\)
therefore from the equiboundedness of
\[
(\|u_n - v_n\|_{L^2(S)} + (r_n)^3 \|\nabla^2 u_n - \nabla^2 v_n\|_{L^2(S)})n
\]
we get
\[
(r_n)^3 \|\nabla u_n - \nabla v_n\|^2_{L^2(S)} \to 0 \quad \text{as } n \to \infty.
\]
Thus (5.10) follows letting $n$ tend to $+\infty$ in (5.12).

STEP 5. For every $u \in GSBV^q(\Omega) \cap L^\infty(\Omega)$ we have
\[
G_p(u, \Omega) \leq \int_{\Omega} g(|\nabla u|) dx + \int_{S_u} \phi(u^+ - u^-) d\mathcal{H}^1 \quad \forall p \geq 1.
\] (5.13)

We start with $u \in W(\Omega)$ (see Subsection 2.3) and, for every $h \in \mathbb{N}$, we consider the sets
\[
B_h := (S_u)_1/h \cap \Omega = \left\{ x \in \Omega : \text{dist}(x, S_u) < \frac{1}{h} \right\};
\]
by the regularity assumptions on $S_u$ we have that $\mathcal{L}^N(B_h) = O(1/h)$ and therefore, setting
\[
\rho_h := h \left( \int_{B_h} |\nabla u|^2 \right)^{1/2},
\]
we have
\[
\lim_{h \to \infty} \frac{1}{\rho_h} \int_{B_h} \nabla u \, dx = 0.
\] (5.14)
By a standard argument based on the use of coarea formula (2.1) (see for example [12]) it is possible to find a sequence $u_h$ satisfying the hypotheses of STEP 2 such that

$$
\|u - u_h\|_{L^\infty(B_h)} \leq \rho_h, \quad \mathcal{H}^N((S_{u_h} \cap B_h) \setminus S_u) \leq \frac{1}{\rho_h} \int_{B_h} |\nabla u| \, dx + O(1). \tag{5.15}
$$

We are going to apply STEP 4 with $A = B_h$, $A' = B_{2h}$, $B = \Omega \setminus B_{3h}$, obtaining for every $K \in \mathbb{N}$ the existence of a cut-off function $\phi^h_K$ such that

$$
G_p(\phi^h_K u_h + (1 - \phi^h_K) u, \Omega) \leq \left(1 + \frac{C}{K}\right) \left(G_p(u, \Omega \setminus S_u) + G_p(u_h, B_h)\right) + C \frac{K^q}{\rho_h} \|u - u_h\|_{L^q(S)}^q + \frac{C}{K} L^N(B_h), \tag{5.16}
$$

where $p \geq q$. By STEP 2 we have

$$
G_p(u_h, B_h) \leq \int_{S_{u_h} \cap B_h} \varphi(u_h^+ - u_h) \, d\mathcal{H}^N \leq \int_{S_u} \varphi(u^+ - u) \, d\mathcal{H}^N + \varphi(2\|u\|_{\infty}) \mathcal{H}^N \left((S_{u_h} \cap B_h) \setminus S_u\right) + \int_{S_u} (\varphi(u_h^+ - u_h) - \varphi(u^+ - u)) \, d\mathcal{H}^N;
$$

using (5.15) and (5.14) and noting that, by the Dominated Convergence Theorem

$$
\lim_{h \to \infty} \int_{S_u} (\varphi(u_h^+ - u_h) - \varphi(u^+ - u)) \, d\mathcal{H}^N = 0,
$$

we therefore obtain

$$
\limsup_{h \to \infty} G_p(u_h, B_h) \leq \int_{S_u} \varphi(u^+ - u) \, d\mathcal{H}^N. \tag{5.17}
$$

Moreover, taking as approximating sequence $u_n = u$ for every $n \in \mathbb{N}$, we discover that

$$
G_p(u, \Omega \setminus S_u) \leq \int_{\Omega} g(|\nabla u|) \, dx; \tag{5.18}
$$

combining (5.17) and (5.18), letting $h \to +\infty$ in (5.16), and taking into account the lower semicontinuity of $G_p$, we finally get (5.13), for $p \geq q$ and therefore for every $p \geq 1$, by virtue of (5.5). For a general $u \in GSJV^q(\Omega) \cap L^p(\Omega)$ we conclude by a standard density argument based on Theorem 2.5.

We are now in a position to conclude the proof of the Theorem. Take $u \in GSJV^q(\Omega) \cap L^p(\Omega)$ and set $u_k := (-k \vee u) \wedge k$; then by (5.13), and the Monotone Convergence Theorem we have

$$
G_p(u, \Omega) \leq \liminf_{h \to \infty} G_p(u_k, \Omega) \leq \lim_{h \to \infty} \int_{\Omega} g(|\nabla u_k|) \, dx + \int_{S_u} \varphi(u_k^+ - u_k) \, d\mathcal{H}^N \leq \int_{\Omega} g(|\nabla u|) \, dx + \int_{S_u} \varphi(u^+ - u) \, d\mathcal{H}^N.
$$

So if $g^\infty(1) \wedge \delta(1) = +\infty$ then we are done; if it is not the case, then the conclusion follows by the fact that, thanks to Theorem 2.3 and an easy truncation argument, $F_{h,q}^N$ coincides with the steady relaxed functional (see
Definition 5.1) of

\[ H(u) := \begin{cases} 
\int_{\Omega} g(|\nabla u|) \, dx + \int_{S_u} \varphi(u^+ - u^-) \, d\mathcal{H}^N & \text{if } u \in GSBV^q(\Omega), \\
+\infty & \text{if } u \in L^1(\Omega) \setminus GSBV^q(\Omega).
\end{cases} \]

The two following corollaries are an immediate consequence of Theorems 5.2, 4.2, 4.12.

**Corollary 5.5** Let \( \Omega \subset \mathbb{R}^N \) be an open bounded set with Lipschitz boundary and let \( f, r, p \) as in Theorem 4.2. Let \( (\varepsilon_n)_n \) be an infinitesimal sequence such that (4.3) holds. If \( b^0(1) = +\infty \) then the functionals

\[ F^N_n(u) = \begin{cases} 
\frac{1}{\varepsilon_n} \int_{\Omega} f(\varepsilon_n u) \, dx + \int_{S_u} \varphi^{(\varepsilon_n)}(u^+ - u^-) \, d\mathcal{H}^N & \text{if } u \in GSBV, \\
+\infty & \text{otherwise in } L^1(\Omega),
\end{cases} \]

\( \Gamma^\varepsilon \)-converge in \( GSBV^q(\Omega) \) to

\[ F^N_n(u) := \begin{cases} 
\int_{\Omega} |\nabla u|^q \, dx + \int_{S_u} \varphi^{(\varepsilon)}(u^+ - u^-) \, d\mathcal{H}^N & \text{if } u \in GSBV, \\
+\infty & \text{otherwise in } L^1(\Omega),
\end{cases} \]

where \( \varphi^{(\varepsilon)} \) is as in Theorem 4.2; conversely, if \( b^0(1) = C \) then the sequence \( F^N_n \) \( \Gamma^\varepsilon \)-converges in \( L^1(\Omega) \) to

\[ F^N(u) := \begin{cases} 
\int_{\Omega} g(|\nabla u|) \, dx + \int_{S_u} \varphi^{(\varepsilon)}(u^+ - u^-) \, d\mathcal{H}^N + C \, D^c u & \text{if } u \in GBV, \\
+\infty & \text{if } u \in L^1(\Omega) \setminus GBV,
\end{cases} \]

with \( \varphi^{(\varepsilon)} \) still given by (3.3) and \( g = (\alpha x^q \land Cx)^{**} \).

**Corollary 5.6** Let \( \Omega \subset \mathbb{R}^N \) be an open bounded set with Lipschitz boundary and let \( f \) be as in Theorem 4.12. Then the family

\[ F^N_n := \begin{cases} 
\frac{1}{\varepsilon} \int_{\Omega} f(\varepsilon |\nabla u|) \, dx + \varepsilon^3 \int_{\Omega} |\nabla^2 u|^2 \, dx & \text{if } u \in W^{2,2}(\Omega), \\
+\infty & \text{otherwise in } L^1(\Omega),
\end{cases} \]

\( \Gamma^\varepsilon \)-converges in \( L^1(\Omega) \) to the functional

\[ F^N(u) := \begin{cases} 
f'(0) \int_{\Omega} |\nabla u| \, dx + \int_{S_u} \varphi(u^+ - u^-) \, d\mathcal{H}^N + f'(0) |D^c u| & \text{if } u \in BV, \\
+\infty & \text{otherwise in } L^1(\Omega),
\end{cases} \]

where \( \varphi \) is the function defined in (3.3) with \( b = f \).

To conclude the \( N \)-dimensional analysis it remains to prove the equicoerciveness of the approximating functionals: this is done in the following proposition.

**Proposition 5.7** Under the same hypotheses of Theorem 5.2, let \( (u_n)_n \subset L^1(\Omega) \) be equiintegrable and such that

\[ \sup_n F^N_n(u_n) < M < +\infty; \]
then \((u_n)\) is strongly precompact in \(L^1(\Omega)\). Suppose in addition that \(F_n^N \rightharpoonup G\) in \(L^1(\Omega)\); then, for every \(g \in L^p(\Omega)\) \((p > 1)\) and \(\beta > 0\), the solutions \(u_n\) of

\[
\min \left\{ F_n^N(v) + \beta \int_\Omega |v - g|^p \, dx : v \in W^{2,2}(\Omega) \right\}
\]

converge, up to subsequences, in the \(L^p(\Omega)\)-norm to a solution of

\[
\min \left\{ G(v) + \beta \int_\Omega |v - g|^p \, dx : v \in L^1(\Omega) \right\}.
\]

**Proof.** As at the beginning of the proof of Theorem 5.2, we fix \(\xi \in S^n\) and we get

\[
M \geq F_n^N(u_n) \geq \int_{\Omega} g_n(y) \, dH^N(\xi),
\]

where \(g_n(y) := F_n((u_n)_\xi^\text{y}, \Omega^\text{y}_\xi)\). Using the equiintegrability assumption, for fixed \(\delta > 0\), we find \(\sigma_\delta > 0\) such that

\[
\mathcal{L}^N(B) \leq \sigma_\delta \Rightarrow \int_B |u_n| \, dx < \delta \quad \forall n \in \mathbb{N}.
\]

Choose \(k > 0\) such that

\[
\frac{M \text{diam}(\Omega)}{k} \leq \sigma_\delta;
\]

set \(A_{n,k} := \{ y \in \Omega : g_n(y) > k \}\) and denote by \(P_\xi\) the orthogonal projection on \(\Pi_\xi\). We now define the new sequence \(v_n\) in the following way

\[
v_n(x) := \begin{cases} 
    u_n(x) & \text{if } P_\xi(x) \in \Omega \setminus A_{n,k} \\
    0 & \text{otherwise}.
\end{cases}
\]

Note that \(\|u_n - v_n\|_{L^1(\Omega)} = \int_{\{x \in \Omega : P_\xi(x) \in A_{n,k}\}} |u_n| \, dx\); since by Chebyshev Inequality, (5.19), and (5.21)

\[
\mathcal{L}^N(\{ x \in \Omega : P_\xi(x) \in A_{n,k} \}) \leq \mathcal{H}^1(A_{n,k}) \text{diam}(\Omega) \leq \frac{\|g_n\|_{L^1(\Omega)}}{k} \text{diam}(\Omega) \leq \frac{M}{k} \text{diam}(\Omega) \leq \sigma_\delta,
\]

recalling (5.20) we finally obtain \(\|u_n - v_n\|_{L^1(\Omega)} \leq \delta\).

Moreover \(F_n((u_n)_\Omega^\text{y}, \Omega^\text{y}_\xi) \leq g_n(y)(1 - \chi_{A_{n,k}}(y)) \leq k\) and therefore, by the one-dimensional results, \((u_n)_\xi^\text{y}\) is precompact in \(L^1(\Omega^\text{y}_\xi)\) for every \(y \in \Omega_\xi\). Since the construction can be repeated for every \(\delta > 0\) and for every \(\xi \in S^n\), the thesis follows by applying Lemma 2.2.

Concerning the second part, we first observe that

\[
\sup_n \left( F_n^N(u_n) + \beta \int_\Omega |u_n - g|^p \, dx \right) \leq \sup_n \lim_{n \to \infty} \left( F_n^N(u_n) + \beta \int_\Omega |u_n - g|^p \, dx \right) \leq \beta \int_\Omega |u - g|^p \, dx < +\infty
\]

and therefore, by the first part of the theorem, there exist \(u \in L^1(\Omega)\) and a subsequence, still denoted by \(u_n\), such that \(u_n \to u\) in \(L^1\). Note that by (5.22) \(\sup_n \|u_n\|_{L^p} < +\infty\) which implies that \(u_n \to u\) weakly in \(L^p\). Since \(F_n^N \rightharpoonup G\), there exists \(v_n \to v\) in \(L^p\) such that \(F_n^N(v_n) \to G(u)\) and therefore, by exploiting the minuality of \(u_n\),

\[
G(u) + \beta \int_\Omega |u - g|^p \, dx \leq \lim_{n \to \infty} \left( F_n^N(u_n) + \beta \int_\Omega |u_n - g|^p \, dx \right) \leq \lim \sup_{n \to \infty} \left( F_n^N(u_n) + \beta \int_\Omega |u_n - g|^p \, dx \right) \leq \lim_{n \to \infty} \left( F_n^N(v_n) + \beta \int_\Omega |v_n - g|^p \, dx \right) = G(u) + \beta \int_\Omega |u - g|^p \, dx
\]
whence,
\[
G(u) + \beta \int_\Omega |u - g|^p \, dx = \lim_{n \to \infty} \left( F^N_n(u_n) + \beta \int_\Omega |u_n - g|^p \, dx \right)
\geq G(u) + \limsup_{n \to \infty} \beta \int_{\Omega} |u_n - g|^p \, dx
\geq G(u) + \liminf_{n \to \infty} \beta \int_{\Omega} |u - g|^p \, dx \geq G(u) + \beta \int_{\Omega} |u - g|^p \, dx.
\]

We deduce \( \int_{\Omega} |u_n - g|^p \, dx \to \int_{\Omega} |u - g|^p \, dx \) and since \( u_n - g \rightharpoonup u - g \) weakly in \( L^p \), we conclude that \( u_n \to u \) in \( L^p \). The minimality of \( u \) follows now from the properties of \( \Gamma \)-convergence. \( \square \)

We conclude the section by remarking that all the examples contained in Section 4 can be generalized to the \( N \)-dimensional case by means of Theorem 5.2. In particular let us underline the following ones.

**Example 5.8 (Perona-Malik Energy)** By Example 4.8 and by Theorem 5.2 we obtain that the functionals

\[
F^N_{\varepsilon}(u) := \begin{cases} 
\frac{1}{\varepsilon} \int_\Omega \log(1 + \varepsilon \alpha |\nabla u|^2) \, dx + \left( \frac{\alpha \varepsilon}{\log \varepsilon} \right)^3 \int_\Omega \| \nabla^2 u \|^2 \, dx & \text{if } u \in W^{2,2}(\Omega), \\
+\infty & \text{otherwise in } L^1(\Omega),
\end{cases}
\]

\( \Gamma^* \)-converge, as \( \varepsilon \to 0^+ \), to

\[
F^N(u) := \begin{cases} 
\alpha \int_\Omega |\nabla u|^2 \, dx + m(0)a^2 \int_{S_u} \sqrt{u^+-u} \, d\mathcal{H}^1 & \text{if } u \in GSBV(\Omega), \\
+\infty & \text{in } L^1(\Omega) \setminus GSBV(\Omega),
\end{cases}
\]
in \( GSBV^2(\Omega) \).

**Example 5.9** Let \( b \) and \( g \) be as in Example 4.13 (and suppose for simplicity \( g(0) = 0 \)); then the family

\[
F^N_{\varepsilon} := \begin{cases} 
\int_\Omega \left( g(|\nabla u|) \wedge \frac{1}{\varepsilon} b(\varepsilon |\nabla u|) \right) \, dx + \varepsilon^3 \int_{\Omega} \| \nabla^2 u \|^2 \, dx & \text{if } u \in W^{2,2}(\Omega), \\
+\infty & \text{otherwise}
\end{cases}
\]

\( \Gamma^* \)-converges in \( L^1(\Omega) \) to the functional

\[
\begin{cases} 
\int_\Omega g(|\nabla u|) \, dx + \int_{S_u} \varphi(u^+-u) \, d\mathcal{H}^1 + C|D^\varepsilon u| & \text{if } u \in GBV(\Omega), \\
+\infty & \text{otherwise},
\end{cases}
\]

where \( \varphi \) is the function associated with \( b \) according to (3.3).

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**References**

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